CHAPTER I

PRELIMINARIES

This chapter presents the basic concepts and theorem of the theory of probability that are needed in the following chapters. The proofs of the statements are omitted; they can be found in the texts of Gnedenko and Kolmogorov [1], Petrov [5] and others.

1. Random Variables, Distribution Functions and Characteristic Functions

Let Ω be a non-empty set of elements. The elements are called points, or elementary events, and will be denoted by the symbol w. The set Ω is called the space of elementary events or the sample space.

Let $\mathfrak{A}$ be a set of subsets of the space Ω, having the properties that

1) $\Omega \in \mathfrak{A}$,

2) if $A \in \mathfrak{A}$ then $\Omega \setminus A \in \mathfrak{A}$,

3) if $A_1, A_2, \ldots$ is a finite or infinite sequence of subsets of Ω belonging to $\mathfrak{A}$, then $\bigcup_n A_n \in \mathfrak{A}$.

The set $\mathfrak{A}$ is called the σ-algebra of events or the Borel field of events, and its elements are called events.

A nonnegative and countably additive function P defined on $\mathfrak{A}$ with $P(\Omega) = 1$, is called a probability measure. The value of

$P(A)$, for each $A \in \mathfrak{A}$, is called the <u>probability of the event A</u>, and the triplet $(\Omega, \mathfrak{A}, P)$ is called a <u>probability space</u>.

Let X be a real-valued function defined on Ω. If the set $X^{-1}(B) = \{w: X(w) \in B\}$ belongs to $\mathfrak{A}$, for any Borel subset B of $\mathbb{R}$, then the function X is called a <u>random variable</u>. For a random variable X, the function P_X induced by X which is defined by

$$(1.1) \qquad P_X(B) = P(\{w: X(w) \in B\}) = P(X^{-1}(B)),$$

for all Borel subsets B of $\mathbb{R}$, is called <u>the probability function of the random variable X</u>. We shall usually use the shorter notation $P(X \in B)$ instead of $P(\{w: X(w) \in B\})$. Hence, the random variable X, defined on a probability space $(\Omega, \mathfrak{A}, P)$, generates a new probability space $(\mathbb{R}, \mathfrak{B}, P_X)$, where $\mathfrak{B}$ is the σ-algebra of Borel sets on the real line $\mathbb{R}$.

Let X be a random variable defined on a probability space. The function F defined on $\mathbb{R}$ by

$$(1.2) \qquad F(x) = P(X \le x),$$

for any real number x, is called the <u>distribution function of the random variable</u> X.

A distribution function F has the following properties: F is non-decreasing, continuous from the right and $F(-\infty) = 0$, $F(+\infty) = 1$. Conversely, an arbitrary function F having the above properties is the distribution function of some random variable defined on some probability space.

If the distribution function F is absolutely continuous then it can be represented as

$$F(x) = \int_{-\infty}^{x} f(t)dt \tag{1.3}$$

for every real number x, where f is a nonnegative function and is integrable on the real line. The function f is known as the <u>density</u> or the <u>probability density function</u> of the corresponding distribution function F.

The random variables $X_1, X_2, \ldots X_n$ are said to be mutually independent or, in short, independent if

$$P\left(\bigcap_{k=1}^{n} \{w: X_k(w) \le x_k\}\right) = \prod_{k=1}^{n} P(X_k \le x_k) \tag{1.4}$$

for any real numbers $x_1, x_2, \ldots, x_n$.

A sequence of random variables $X_1, X_2, \ldots$ defined on the same probability space is said to be <u>a sequence of independent random variables</u> if the $X_1, X_2, \ldots, X_n$ are mutually independent for every n. For an arbitrary sequence of distribution functions $F_1, F_2, \ldots$ there exists a probability space $(\Omega, \mathfrak{A}, P)$ and a sequence of independent random variables $X_1, X_2, \ldots$ defined on it such that for every n the distribution function of X_n is F_n.

If $X_1, X_2, \ldots, X_n$ are independent and ϕ is a Borel measurable function defined on the real line then $\phi(X_1), \phi(X_2), \ldots, \phi(X_n)$ are also independent random variables, and if f and h are Borel measurable functions with values in $\mathbb{R}$ defined on $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$, respectively, where $m < n$ then $g(X_1, X_2, \ldots, X_m)$ and $h(X_{m+1}, X_{m+2}, \ldots, X_n)$ are independent.

If the random variables X_1 and X_2 are independent and have distributions F_1 and F_2, respectively, then the sum $X_1 + X_2$ has the distribution function F, where

$$(1.5) \qquad F(x) = \int_{-\infty}^{\infty} F_1(x-y)dF_2(y).$$

The integral on the right side is said to be the <u>convolution</u> or the <u>composition</u> of the distributions F_1 and F_2 and is denoted by $F_1 * F_2$.

Let F be a distribution function and t a real number. The function φ defined by

$$(1.6) \qquad \varphi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x) ,$$

is called the <u>characteristic function of the distribution</u> F. If the random variable X has distribution function F, then φ is also said to be the <u>characteristic function of X</u>.

It follows from the above definition that every characteristic function φ satisfies: $\varphi(0) = 1$, $|\varphi(t)| \leq 1$ for every real t and φ is uniformly continuous on the real line. Furthermore, $\varphi(-t) = \overline{\varphi(t)}$, the complex conjugate of $\varphi(t)$. If φ is the characteristic function of the random variable X then the function ψ, defined for any real t by

$$(1.7) \qquad \psi(t) = e^{ibt}\, \varphi(at) ,$$

is the characteristic function of $aX + b$, where a and b are real constants. If the random variables $X_1, X_2, \ldots, X_n$ are independent and have characteristic functions $\varphi_1, \varphi_2, \ldots, \varphi_n$, respectively, then the characteristic function φ of the sum $X_1 + X_2 + \ldots + X_n$ is given by

$$\varphi(t) = \varphi_1(t) \cdot \varphi_2(t) \cdot \ldots \cdot \varphi_n(t). \tag{1.8}$$

The correspondence between the distribution functions and the characteristic function is one-to-one. Moreover, if a sequence of distribution functions $\{F_n\}$ converges to a distribution function F at every point of continuity of F, then the corresponding sequence of characteristic functions $\{\varphi_n\}$ converges to the characteristic function φ of F and the converse is true.

2. Infinitely Divisible Distribution Functions

A random variable X is said to be *infinitely divisible* if, for every natural number n, it can be represented as the sum

$$X = X_{n1} + X_{n2} + \ldots + X_{nn} \tag{2.1}$$

of n independent identically distributed random variables $X_{n1}, X_{n2}, \ldots, X_{nn}$. The distribution function and the corresponding characteristic function of an infinitely divisible random variable are also said to be *infinitely divisible*.

From the above definition, it follows that X is infinitely divisible if and only if its characteristic function φ is, for every natural number n, the n-th power of a characteristic function φ_n

(which, of course, depends on n), that is

$$(2.2) \qquad \varphi(t) = [\varphi_n(t)]^n.$$

Presented here are some important properties of infinitely divisible characteristic function (c.f. [5])

1) if φ is an infinitely divisible characteristic function then $\varphi(t) \neq 0$ for all t,
2) if φ_1 and φ_2 are infinitely divisible characteristic functions then $\varphi_1 . \varphi_2$, the multiplication of φ_1 and φ_2, is also an infinitely divisible characteristic function and
3) if $\{\varphi_n\}$ is a sequence of infinitely divisible characteristic functions which converges to some characteristic function φ, then φ is also infinitely divisible.

It is well known (c.f. [1] and [5]) that the characteristic function φ is infinitely divisible if and only if its logarithm can be represented in the form

$$(2.3) \qquad \log \varphi(t) = \gamma it + \int_{-\infty}^{\infty} f(t,u)dG(u) \quad ,$$

where γ is a real constant, f(t,u) is given by

$$(2.4) \qquad f(t,u) = \begin{cases} (e^{itu} - 1 - \frac{itu}{1+u^2}) . \frac{1+u^2}{u^2} & , \quad u \neq 0 \\ -\frac{t^2}{2} & , \quad u = 0 \; , \end{cases}$$

G(u) is a bounded nondecreasing function which is continuous from the right and $G(-\infty) = 0$.

The representation of φ given by (2.3) is unique, and it is known as the formula of Lévy and Khintchine.

There is another representation of the logarithm of an infinitely divisible characteristic function φ, known as Lévy's formula :

$$(2.5) \quad \log \varphi(t) = \gamma it - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0^-} (e^{itx} - 1 - \frac{itx}{1+x^2}) d\, M(x) + \int_{0^+}^{+\infty} (e^{itx} - 1 - \frac{itx}{1+x^2}) d\, N(x),$$

where γ, $\sigma^2 > 0$ are constants, $M(x)$ and $N(x)$ are nondecreasing functions defined on $(-\infty,0)$ and $(0,+\infty)$ respectively, $M(-\infty) = 0$, $N(+\infty) = 0$ and

$$\int_{-\varepsilon}^{0^-} x^2 d\, M(x) + \int_{0^+}^{\varepsilon} x^2 d\, N(x) \qquad < +\infty$$

for every finite positive real number ε.

The characteristic function φ is infinitely divisible if and only if its logarithm can be represented by Levy's formula (2.5). Representation (2.5) of an infinitely divisible characteristic function is unique.

For an infinitely divisible characteristic function φ, representations (2.3) and (2.5) are related by

$$M(x) = \int_{-\infty}^{x} \frac{1+u^2}{u^2} d\, G(u) \quad , \qquad \text{for } x < 0$$

$$N(x) = - \int_{x}^{+\infty} \frac{1+u^2}{u^2} \, d\,G(u) \quad , \qquad \text{for } x > 0$$

and $$\sigma^2 = G(0+) - G(0-)$$

(c.f. [5], p.31).

3. Stable Laws.

A distribution function F is said to be stable if, for all real numbers a_1, b_1, a_2, b_2 such that a_1 and a_2 are positive, there correspond constants $a > 0$ and b such that the equation

$$(3.1) \qquad F(a_1x+b_1) * F(a_2x+b_2) = F(ax+b) \quad .$$

holds, where * denotes the convolution operation.

In order that a distribution function F be stable, it is necessary and sufficient that the logarithm of its characteristic function be representable by the formula

$$(3.2) \qquad \log \varphi(t) = i\gamma t + c|t|^{\alpha}\{1 + \frac{i\beta t}{|t|} w(t,\alpha)\} \quad ,$$

where α, β, γ, c are constants (γ is any real number, $-1 \leq \beta \leq 1$, $0 < \alpha \leq 2$, $c \geq 0$) and

$$w(t,\alpha) = \begin{cases} \tan \frac{\pi}{2} \alpha & , \text{ for } \alpha \neq 1 \\ \frac{2}{\pi} \log |t| & , \text{ for } \alpha = 1 \end{cases}$$

(c.f. [1], p. 162-164).

The constant α in formula (3.2) is known as the characteristic exponent of the stable law.

It is known that every stable law is infinitely divisible where the functions M, N and the constant σ^2 in Lévy's formula (2.5) are given as follows:

$$\text{for } 0 < \alpha < 2, \quad M(x) = \frac{c_1}{|x|^\alpha}, \quad N(x) = -\frac{c_2}{x^\alpha} \quad \text{and}$$

$$\sigma = 0, \quad \text{and}$$

for $\alpha = 2$, $M(x) = N(x) = 0$ and σ is nonnegative (where the constants c_1 and c_2 are both nonnegative with $c_1 + c_2 > 0$).

We see, therefore, that the normal distribution with parameter (a,σ), whose characteristic function is in the form

$$\varphi(t) = e^{ita-\sigma^2\frac{t^2}{2}},$$

is a stable law with characteristic exponent 2, and the Cauchy distribution with parameters λ and ν, whose density function is

$$p(x) = \frac{\nu}{\pi} \frac{1}{\nu + (x-\lambda)^2}, \quad x \in R, \ \lambda \in R, \ \nu > 0,$$

is a stable law with characteristic exponent 1.

4. Some General Limit Theorems.

In this section we state some important theorems, which are criteria for the existence of a limit distribution for sums of independent random variables.

Consider a double sequence

$$\begin{array}{ll} & \xi_{11}, \xi_{12}, \ldots, \xi_{1k_1} \\ & \xi_{21}, \xi_{22}, \ldots, \xi_{2k_2} \\ (4.1) & \vdots \\ & \xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n} \\ & \ldots\ldots\ldots, \end{array}$$

of random variables which are independent in each row and $k_n \to \infty$ as $n \to \infty$, where $\{\xi_{nk}\}$ is <u>infinitesimal</u>, that is,

$$(4.2) \qquad \lim_{n\to\infty} \sup_{1\le k\le k_n} P(|\xi_{nk}| \ge \varepsilon) = 0$$

for every positive real number ε.

It is known, since the assertion was proved by A.Ya. Khintchine in 1937, that the class of limit distribution functions of the sums

$$(4.3) \qquad \xi_{n1} + \xi_{n2} + \ldots + \xi_{nk_n} - A_n ,$$

for some suitably chosen sequence of constants $\{A_n\}$, is the class of infinitely divisible distribution functions (c.f. [1] p. 115-116).

In studying the convergence of distribution functions of sums of the form (4.3), some necessary and sufficient conditions were found. Here is a general theorem for sums of independent random variables and the proof can be found in [1].

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<u>Theorem 1</u> The sums

$$\xi_{n1} + \xi_{n2} + \dots + \xi_{nk_n} - A_n$$

for some suitable constants A_n, converge to a limit if and only if there exist non-decreasing functions $M(x)$ and $N(x)$, defined on the intervals $(-\infty,0)$ and $(0,+\infty)$ respectively with $M(-\infty) = 0$ and $N(+\infty) = 0$, and a constant $\sigma \geq 0$ such that

$$\text{(4.4)}\quad \begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{k_n} F_{nk}(x) &= M(x) \quad , \quad \text{for } x < 0 , \\ \lim_{n\to\infty} \sum_{k=1}^{k_n} [F_{nk}(x)-1] &= N(x) \quad , \quad \text{for } x > 0 , \end{aligned}$$

at every point of continuity of $M(x)$ and $N(x)$, and

$$\text{(4.5)}\quad \lim_{\varepsilon\to 0} \underline{\lim}_{n\to\infty} \sum_{k=1}^{k_n} \{ \int_{|x|<\varepsilon} x^2 dF_{nk}(x) - (\int_{|x|<\varepsilon} x \, dF_{nk}(x))^2 \}$$

$$= \lim_{\varepsilon\to 0} \overline{\lim}_{n\to\infty} \sum_{k=1}^{k_n} \{ \int_{|x|<\varepsilon} x^2 dF_{nk}(x) - (\int_{|x|<\varepsilon} x dF_{nk}(x))^2 \} = \sigma^2 ,$$

where $F_{nk}(x)$ denotes the distribution function of ξ_{nk}.

As a consequence of the above theorem, when the limit distribution is the normal law, we have the following theorems. Their proofs are not presented here, they can be found in most of the texts dealing with the theory of sums of independent random variables.

<u>Theorem 2</u> For some suitably chosen constants A_n, the distributions of the sums

$$\xi_{n1} + \xi_{n2} + \dots + \xi_{nk_n} - A_n$$

converge to the normal law,

$$(4.6) \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz ,$$

and the summand $\{\xi_{nk}\}$, $1 \le k \le k_n$, are infinitesimal if and only if the conditions

$$(4.7) \qquad \lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|\ge\varepsilon} dF_{nk}(x) = 0 \qquad \text{and}$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \{ \int_{|x|<\varepsilon} x^2 dF_{nk}(x) - (\int_{|x|<\varepsilon} x dF_{nk}(x))^2 \} = 1$$

are satisfied for every positive real number ε, where F_{nk} is the distribution function of ξ_{nk}.

<u>Theorem 3</u> Let $X_1, X_2, \dots$ be a sequence of independent random variables. Then there exist real constants A_n and $B_n > 0$ such that the distribution of

$$(4.8) \qquad \frac{X_1 + X_2 + \dots + X_n}{B_n} - A_n$$

converges to the normal law (4.6), and the summands

$$\xi_{nk} = \frac{X_k}{B_n} \qquad , \quad 1 \le k \le n$$

satisfy the condition of being infinitesimal if and only if there exists a sequence of constants c_n such that $\lim_{n\to\infty} c_n = \infty$, (4.9); $$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x|>c_n} dF_k(x) = 0 \quad \text{and}$$

(4.10); $$\lim_{n\to\infty} \frac{1}{c_n^2} \sum_{k=1}^{n} \left\{ \int_{|x|<c_n} x^2 dF_k(x) - \left(\int_{|x|<c_n} x dF_k(x) \right)^2 \right\} = +\infty .$$

When a sequence $\{c_n\}$, with the indicated properties, exists, the constants A_n and B_n in (4.8) can be chosen by

$$B_n^2 = \sum_{k=1}^{n} \left\{ \int_{|x|<c_n} x^2 dF_k(x) - \left(\int_{|x|<c_n} x dF_k(x) \right)^2 \right\} \quad \text{and}$$

(4,12),

$$A_n = \frac{1}{B_n} \sum_{k=1}^{n} \int_{|x|<c_n} x dF_k(x) .$$ The proof of this assertion can be found in [5].

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