สมการเชิงฟังก์ชันกำลังสองทางเลือกบนอาบีเลียนกรุปที่หารด้วยสองได้

นางสาวเจนจิรา ทิพย์ญาณ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์และวิทยาการคณนา ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2558

ALTERNATIVE QUADRATIC FUNCTIONAL EQUATION ON 2-DIVISIBLE ABELIAN GROUPS

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A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Applied Mathematics and Computational Science
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2015

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ACKNOWLEDGEMENTS

The dissertation leading to the completion of this Ph.D. would have not been this successful without the opportunity and blessing support from my advisor, Associate Professor Dr. Paisan Nakmahachalasint. Other than being a wonderful academic mentor, he has greatly inspired me by his thoughts that have enriched my personal life. My deep gratitude goes to Assistant Professor Dr. Nataphan Kitisin, Assistant Professor Dr. Khamron Mekchay, Assistant Professor Dr. Keng Wiboonton and Dr. Tippaporn EungrasameeAssociate for their recommendations and contributions to the research. I would like to express my indebted appreciation to Associate Professor Dr. Patanee Udomkavanich, whom I admire as my role model in Mathematics Teaching. Associate Professor Dr. Patanee Udomkavanich is always merciful and positively engaged in providing knowledge and meaningful living. Many thanks to every my teacher in Department of Mathematics and Computer Science for teaching and mentoring. My profound thanks to my fellow students for sharing their cutting-edge knowledge and great minds. I must also thank Khiansa Pittayakhom School for allowing me to pursue my Ph.D.

Finally, I would like to dedicate this dissertation to my beloved family. They have been strongly supportive even in my tough time. Throughout the duration of my study, I am particularly proud to be a part of my family and Chulalongkorn University.

CONTENTS

CHAPTER I
INTRODUCTION

"***Functional equations***" originally are equations in which the unknown (or unknowns) is function. J. Aczél [2] described functional equations as follows.

> Functional equations are equations, both sides of which are terms constructed from a finite number of unknown functions (of a finite number of variables) and from a finite number of independent variables. This construction is effected by a finite number of known functions of one or several variables (including the four species) and by finitely many substitutions of terms which contain known and unknown functions into other known and unknown functions. The functional equations determine the unknown functions. We speak of functional equations or system of functional equations, depending on whether we have one or several equations.

According J. Aczél, J.G. Dhombres [1] and M. Kuczma [9], we know that the history of the study functional equation may be dated back more than 2200 years when Archimedes made use of recurrences.

The beginning of a *theory of functional equations* related to the work of J. Aczél, a Hungarian mathematician, who was an excellent specialist in this field(see [2]). Theory of functional equations then has been developed gradually. Especially in the last two decades, it has grown rapidly. A lot of mathematical papers investigating functional equations have been published. Moreover, functional equations are included in the mathematical olympiad contest so student could widely learn in this branch of mathematics. Now functional equations become an important research field with, a number of interesting results and several applications.

Probably the best known and the most basic functional equation is the *Cauchy functional equation*

$$f(x+y) = f(x) + f(y)$$

containing two variables x, y and one unknown function f of variable. There are certain other functional equations which can be transformed into the Cauchy functional equation. Three most important such functional equations are

$$\begin{aligned} f(x+y) &= f(x)f(y) && \text{(exponential)} \\ f(xy) &= f(x)+f(y) && \text{(logarithmic)} \\ f(xy) &= f(x)f(y) && \text{(multiplicative).} \end{aligned}$$

Solving a functional equation is always the art that many mathematicians are crazy, investigating the functional inequalities becomes more interesting problem and determining the general solution of an alternative functional equation is a challenging problem.

Next, we will give some general background of functional equation and stability. Then we threat an alternative functional equation and its literature. Furthermore we refer to the motivation of our proposed problem.

1.1 Functional Equations

At first we will give a definition of a functional equation based on the concept of the terms along which was given in the book of Kuczma [9].

Definition 1.1 (Kuczma)**.** A *term* is defined by the following conditions:

1. Independent variables are terms.
2. If $t_1, \ldots, t_p$ are terms and $f(x_1, \ldots, x_p)$ is a function of p variables, then $f(t_1, \ldots, t_p)$ also is a term.
3. There exist no other terms.

Then a functional equation may be defined as follows:

Definition 1.2 (Kuczma)**.** A *functional equation* is an equality $t_1 = t_2$ between two terms t_1 and t_2 which contain at least one unknown function and a finite number of independent variables. This equality is to be satisfied identically with respect to all the occurring variables in a certain set (of any sort).

The notion of a functional equation as defined above does not contain differential, integral, operator equation and generally equation in which infinitesimal operations are performed. As we know, the solutions of a functional equation must be functions.

A function satisfying a functional equation on a given domain is called a *solution* of the equation on that domain. Next, we will give some examples of functional equations.

Example 1.3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(x) + y \quad \text{for all } x, y \in \mathbb{R}. \tag{1.1}$$

Solution. Assume that there exits a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1).
Setting $x = 0$ in (1.1). , then

$$f(y) = y + f(0) \quad \text{for all } y \in \mathbb{R}.$$

which implies that the function f must be given by $f(x) = x + c$, where c is a constant.

On the other hand, if a function f is defined by $f(x) = x + c$ for all $x \in \mathbb{R}$ where c is a constant, then we have

$$f(x+y) = x + y + c = (x+c) + y = f(x) + y \quad \text{for all} \quad x, y \in \mathbb{R}.$$

□

In general, a functional equation may not necessarily have a solution. The next example shows such an example.

Example 1.4. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(y) + x + 1 \quad \text{for all } x, y \in \mathbb{R}. \tag{1.2}$$

Solution. Suppose that there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.2). Substituting $y = 0$ into (1.2), we obtain

$$f(x) = f(0) + x + 1 \quad \text{for all } x \in \mathbb{R}.$$

which implies that the function f must be given by $f(x) = x + c$, where c is a constant.

Conversely, if a function f is given by $f(x) = x + c$ for all $x \in \mathbb{R}$, then we see that the left-hand side of (1.2) becomes

$$f(x+y) = x + y + c$$

while the right-hand side of (1.2) is

$$f(y) + x + 1 = x + y + c + 1.$$

Since $c \neq c + 1$, there is no function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.2). $\square$

A classical example of functional equation is the *Cauchy functional equation* given as follows:

$$f(x+y) = f(x) + f(y) \tag{C}$$

for all $x, y, x+y$ in the domain of f. In 1821, A.L. Cauchy [4] proved that all continuous solutions $f : \mathbb{R} \to \mathbb{R}$ are linear functions given by $f(x) = cx$ for all $x \in \mathbb{R}$ where c is a constant.

The simplest and most elegant variation of the Cauchy functional equation is the Jensen's functional equation which may be expressed in the form

$$f\Big(\frac{x+y}{2}\Big) = \frac{f(x) + f(y)}{2}. \tag{J}$$

Every solution of Jensen's functional equation will be called *Jensen function.*The continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of (J) are $f(x) = a + bx$ for all $x \in \mathbb{R}$ where a and

b are constants.

The so-called *quadratic functional equation* is the equation of the form

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{Q}$$

The continuous solution to the equation (Q) on $\mathbb{R}$ is of the form $f(x)=cx^2$ for all $x\in\mathbb{R}$. Moreover, every solution of the equation (Q) will be called a *quadratic function*.

In the next section, we shall provide the history of the study of stability problems which are now popular research problems for many mathematicians (please refer to [5] and [14] for details).

1.2 The Stability

The problem of stability originated from the question of S.M. Ulam [17] in 1940. He gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of homomorphism :

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$ Given $\epsilon>0$, does there exist a $\delta>0$ such that if a function $h:G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$?

If the answer to this question is affirmative, we say that the functional equation $h(xy)=h(x)h(y)$ is *stable*. The first answer to this question was given by D.H. Hyers [8] in 1941 as follows.

Theorem 1.5. *(D.H. Hyers) Assume that E_1 and E_2 are Banach spaces. If a function $f:E_1\to E_2$ satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon$$

for some $\varepsilon \geq 0$ *and for all* $x, y \in E_1$*, then the limit*

$$a(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each x *in* E_1 *and* $a : E_1 \to E_2$ *is unique additive function such that*

$$\|f(x) - a(x)\| \leq \varepsilon$$

for any $x \in E_1$*. Moreover, if* $f(tx)$ *is continuous in* t *for each fixed* $x \in E_1$*, then* a *is linear.*

This result marks the staring point of the theory of Hyers-Ulam stability of functional equations.

Later, T. Aoki [3] and Th. M. Rassias [13] generalized the concept of the Hyers-Ulam stability which propelled many mathematicians to study this kind of stability for a number of important functional equations. Rassias' result is given in the following theorem.

Theorem 1.6. *(Th.M. Rassias) Let* $f : E_1 \to E_2$ *be a mapping between Banach spaces* E_1, E_2 *such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \quad \textit{for all} \quad x, y \in E_1,$$

for some constants $\theta > 0$ *and* $0 \leq p < 1$*. Then there exists a unique additive mapping* $A : E_1 \to E_2$ *such that*

$$\|A(x) - f(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p \quad \textit{for all} \quad x \in E_1$$

Moreover, if $f(tx)$ *is continuous in* t *for each fixed* $x \in E_1$*, then* A *is linear.*

Now we can briefly say that a study of stability of a functional equation is to consider functional inequality and ask: *does there exist a solution of the functional equation which approximates the solutions of the inequality within a given distance?*

In the next section, we provide the study of an alternative functional equation which is a challenging research problems for many mathematicians.

1.3 Alternative functional equations

Normally, function equation has only one equation which determine a relativity of function for any set of independent variable such as, (C) when we determined x, y, we will know a visible relativity, that is $f(x+y) = f(x) + f(y)$ but, in the alternative functional equation, a set of independent variable possibly have more relativity for the example, alternative Cauchy functional equation

$$f(x+y) = f(x) + f(y) \quad \text{or} \quad f(x+y) = -f(x) - f(y).$$

The difference from primary thing is, when we determine x, y we will have two choices for choose, $f(x+y) = f(x) + f(y)$ or $f(x+y) = -f(x) - f(y)$ but, we can't know any equation is true (or all of these true, if a value is suitably in any positions). The challenges of alternative function is *"we don't know one of these true or both either."* And in addition, when focus on two set of independent variable we maybe choose different alternative for that, such an alternative Cauchy functional equation in primary, if we knew $f(4) = f(3) + f(1)$ maybe possible in $f(8) = -f(6) - f(2)$ for more detail, please see ([11]).

Next sections we introduce an alternative Cauchy functional equation, an alternative quadratic functional equation and some examples.

1.3.1 Alternative Cauchy functional equations

An alternative Cauchy functional equation derived from (C) may read

$$f(x) + f(y) = \pm f(x+y) \tag{1.3}$$

which its general solution for functions defined on semigroups has been discussed by M. Kuczma [10]. A more general alternative Cauchy functional equation of the form

$$[f(x+y) - af(x) - bf(y)] \cdot [f(x+y) - cf(x) - df(y)] = 0,$$

where f is a function defined on a commutative group, has been solved by R. Ger [7]. Next example shows that the alternative Cauchy functional equation (1.3) is equivalent to the Cauchy functional equation (C).

Example 1.7. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x+y) = \pm(f(x)+f(y)) \quad \text{for all} \quad x, y \in \mathbb{R}. \tag{1.4}$$

if and only if f satisfies

$$f(x+y) = f(x)+f(y) \quad \text{for all} \quad x, y \in \mathbb{R}. \tag{1.5}$$

Proof. It is clear that if f satisfies (1.5), then f satisfies(1.4). Thus it is sufficient to prove that if f satisfies (1.4), then f satisfies (1.5). Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = \pm(f(x)+f(y))$. Setting $(x, y) = (0, 0)$ in (1.4), we have $f(0) = 0$. Replacing $y = -x$ into (1.4) and using $f(0) = 0$ give $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Next, we will prove that f satisfies (1.5) for all $x, y \in \mathbb{R}$ by contradiction. Assume that there exist $x_0, y_0 \in \mathbb{R}$ such that

$$f(x_0+y_0) \neq f(x_0)+f(y_0). \tag{1.6}$$

Since $f(x_0+y_0) = \pm f(x_0)+f(y_0)$, we are left with

$$f(x_0+y_0) = -f(x_0)-f(y_0). \tag{1.7}$$

Setting $(x, y) = (x_0+y_0, -y_0)$ in (1.4) and using $f(-x) = -f(x)$, we get

$$f(x_0) = f(x_0+y_0)-f(y_0) \quad \text{or} \quad f(x_0) = -f(x_0+y_0)+f(y_0). \tag{1.8}$$

Since $f(x_0+y_0) \neq f(x_0)+f(y_0)$, we have

$$f(x_0+y_0) = -f(x_0)+f(y_0). \tag{1.9}$$

Setting $(x, y) = (x_0+y_0, -x_0)$ in (1.4) and using $f(-x) = -f(x)$, we get

$$f(y_0) = f(x_0+y_0)-f(x_0) \quad \text{or} \quad f(y_0) = -f(x_0+y_0)+f(x_0). \tag{1.10}$$

Since $f(x_0+y_0) \neq f(x_0)+f(y_0)$, we have

$$f(x_0+y_0) = f(x_0)-f(y_0). \tag{1.11}$$

Considering (1.9), (1.11) and (1.7) give $f(x_0) = f(y_0) = f(x_0 + y_0) = 0$. Then $f(x_0+y_0) = f(x_0)+f(y_0)$, which is a contradiction with (1.6). Therefore, $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. □

Next example, we show that another alternative Cauchy functional equation $f(x + y) = f(x) \pm f(y)$ is equivalent to (1.5) on 2-divisible abelian groups. Let $(G, +)$ be a 2-divisible group, i.e., for every $x \in G$, there exists $y \in G$ such that $x = 2y$ and $(Y, +)$ be a real (or rational or complex) linear space.

Example 1.8. A function $f : G \to Y$ satisfies

$$f(x + y) = f(x) \pm f(y) \quad \text{for all} \quad x, y \in G. \tag{1.12}$$

if and only if f satisfies

$$f(x + y) = f(x) + f(y) \quad \text{for all} \quad x, y \in G. \tag{1.13}$$

Proof. Suppose that $f : G \to Y$ satisfies (1.12) for all $x, y \in G$. To show that $f(2x) = 2f(x)$ for all $x, y \in G$. Setting $y = x$ in (1.12) gives

$$f(2x) = 2f(x) \quad \text{or} \quad f(2x) = 0 \quad \text{for all} \quad x \in G. \tag{1.14}$$

Assume that there exists $x_0 \in G$ such that $f(2x_0) \neq 2f(x_0)$. Since G is a 2-divisible group, we let $z_0 \in G$ such that $2z_0 = x_0$. Therefore,

$$f(4z_0) \neq 2f(2z_0). \tag{1.15}$$

Replacing $x = 2z_0$ into (1.14) and using the assumption (1.15), we have

$$f(4z_0) = 0. \tag{1.16}$$

From (1.15) and (1.16), we can see that

$$f(2z_0) \neq 0. \tag{1.17}$$

Setting $x = z_0$ in (1.14) and from (1.17), we get

$$f(2z_0) = 2f(z_0). \tag{1.18}$$

Replacing $(x, y) = (2z_0, z_0)$ into (1.12) and using (1.18), we obtain

$$f(3z_0) = 3f(z_0) \quad \text{or} \quad f(3z_0) = f(z_0). \tag{1.19}$$

Plugging $(x, y) = (3z_0, z_0)$ in (1.12) and using (1.16), we obtain

$$f(3z_0) = -f(z_0) \quad \text{or} \quad f(3z_0) = f(z_0). \tag{1.20}$$

Considering (1.19) and (1.20), we have

$$f(3z_0) = f(z_0). \tag{1.21}$$

Setting $(x, y) = (4z_0, z_0)$ in (1.12) and using (1.16), we obtain

$$f(5z_0) = f(z_0) \quad \text{or} \quad f(5z_0) = -f(z_0). \tag{1.22}$$

Putting $(x, y) = (3z_0, 2z_0)$ in (1.12), then using (1.18) and (1.21), we obtain

$$f(5z_0) = 3f(z_0) \quad \text{or} \quad f(5z_0) = -f(z_0). \tag{1.23}$$

Considering (1.22) and (1.23) gives

$$f(5z_0) = -f(z_0). \tag{1.24}$$

Setting $x = 3z_0$ in (1.14) and using (1.21), we get

$$f(6z_0) = 2f(z_0) \quad \text{or} \quad f(6z_0) = 0. \tag{1.25}$$

Putting $(x, y) = (4z_0, 2z_0)$ in (1.12), then using (1.18) and (1.16), we obtain

$$f(6z_0) = 2f(z_0) \quad \text{or} \quad f(6z_0) = -2f(z_0). \tag{1.26}$$

Plugging $(x, y) = (5z_0, z_0)$ in (1.12) and using (1.24), we obtain

$$f(6z_0) = 0 \quad \text{or} \quad f(6z_0) = -2f(z_0). \tag{1.27}$$

Comparing (1.25), (1.26) and (1.27), we conclude that $f(z_0) = 0$, which contradicts (1.17) and (1.18). Therefore, $f(2x) = 2f(x)$ for all $x \in G$. Next, we shall show that (1.12) is equivalence to (1.13). The sufficiency of this example is obvious. For the necessity, suppose there exist $x_0, y_0 \in G$ such that

$$f(x_0 + y_0) \neq f(x_0) + f(y_0). \tag{1.28}$$

Setting $(x, y) = (x_0, y_0)$ into (1.12) and using (1.28), we obtain

$$f(x_0 + y_0) = f(x_0) - f(y_0). \tag{1.29}$$

Replacing $(x, y) = (y_0, x_0)$ into (1.12) and using (1.28) gives

$$f(x_0 + y_0) = f(y_0) - f(x_0). \tag{1.30}$$

Considering (1.29) and (1.30), we conclude that

$$f(x_0) = f(y_0) \quad \text{and} \quad f(x_0 + y_0) = 0. \tag{1.31}$$

Setting $(x, y) = (x_0 - y_0, x_0 + y_0)$ into (1.12), then using (1.31) and $f(2x) = 2f(x)$ for all $x \in G$ give

$$f(x_0 - y_0) = 2f(x_0) \tag{1.32}$$

Putting $(x, y) = (y_0, x_0 - y_0)$ in (1.12) and using (1.32), we infer that $f(x_0) = 0$. From (1.31), we get $f(x_0 + y_0) - f(x_0) - f(y_0) = 0$, a contradiction. Therefore, $f(x + y) = f(x) + f(y)$ for all $x, y \in G$. $\square$

1.3.2 Alternative quadratic functional equations

In 1995, F. Skof [15] proposed the following four alternative quadratic functional equations:

$$|f(x + y)| = |2f(x) + 2f(y) - f(x - y)|, \tag{1.33}$$

$$|f(x - y)| = |2f(x) + 2f(y) - f(x + y)|, \tag{1.34}$$

$$|2f(y)| = |f(x + y) + f(x - y) - 2f(x)|, \tag{1.35}$$

$$\text{and} \quad |2f(x)| = |f(x + y) + f(x - y) - 2f(y)| \tag{1.36}$$

and proved that for the class of functions $f : X \to \mathbb{R}$, where X is a real linear space, each of the above functional equation is equivalent to the quadratic functional equation (Q). Nevertheless, the alternative quadratic functional equations:

$$|f(x+y)+f(x-y)| = |2f(x)+2f(y)| \tag{1.37}$$

is considerably subtle, and it can only be proved that f is rationally homogeneous of degree 2, i.e., $f(rx) = r^2 f(x)$ for all rational numbers r and for all $x \in X$. But for a specific case when $X = \mathbb{R}$ and f is a continuous function, it can be successfully shown that the alternative quadratic functional equation (1.37) is equivalent to the quadratic functional equation (Q).

A more recent result concerning an alternative quadratic functional equation is due to G.L. Forti [6] who studied the solution of the following functional equation:

$$f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \in \{0, 1\},$$

where f is a function from a group $(G, \cdot)$ to $\mathbb{R}$ and G possesses certain additional properties. In 2015, P. Nakmahachalasint [12] has shown that the alternative quadratic functinal equation $f(xy^{-1}) + f(xy) = \pm 2(f(x) + f(y))$ is equivalent to the quadratic functional equation $f(xy^{-1}) + f(xy) = 2f(x) + 2f(y)$ when f is a mapping from 2-divisible group $(G, \cdot)$ to a uniquely divisible abelian group $(G^*, +)$. However, an alternative quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) \pm 2f(y)$$

has not been investigated. Therefore, in this dissertation, we will investigate the solution of the alternative quadratic functional equation on 2-divisible abelian group.

1.4 Proposed Problem

Let $(G, +)$ be a 2-divisible abelian group and let $(Y, +)$ be a real (or rational or complex) linear space.

In this thesis, we will prove that the alternative quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)\pm 2f(y) \quad \text{for all} \quad x,y\in G$$

is equivalent to the classical quadratic functional equation (Q).

In Chapter II, we introduced an n-dimensional Jensen type functional equation

$$\sum_{i=1}^{n} r_i f(x_i) = f\left(\sum_{i=1}^{n} r_i x_i\right). \tag{NJ}$$

We give its general solution and investigate its stabilities of various types.

In Chapter III, we prove the equivalence of the classical quadratic functional equation and an alternative quadratic functional equation on 2-divisible abelian groups.

CHAPTER II
THE GENERALIZED STABILITY OF AN n-DIMENSIONAL JENSEN TYPE FUNCTIONAL EQUATION

Results in this section was published in Thai Journal of Mathematics ([16]). Throughout this chapter, we let $n > 1$ be an integer, $r_1, r_2, \ldots, r_n$ be positive rational numbers satisfying

$$\sum_{i=1}^{n} r_i = 1. \tag{2.1}$$

Here, we study an n-dimensional Jensen type functional equation

$$\sum_{i=1}^{n} r_i f(x_i) = f\left(\sum_{i=1}^{n} r_i x_i\right) \tag{NJ}$$

where f is a function from a real (or rational or complex) linear space X to a real (or rational or complex) linear space Y.

Next theorem gives the general solutions for (NJ).

Theorem 2.1. *Let $f : X \to Y$ where X and Y are real (or rational or complex) linear spaces. Then f satisfies (NJ) for all $x_1, \ldots x_n \in X$ if and only if $f(x) = A(x) + k$ for all $x \in X$ where $A : X \to Y$ is an additive function and k is a constant.*

Proof. (Neccessity) Suppose $f : X \to Y$ satisfies the functional equation (NJ). Define a function $g : X \to Y$ by

$$g(x) = f(x) - f(0)$$

for all $x \in X$. Note that $g(0) = 0$.

$$
\begin{aligned}
g\left(\sum_{i=1}^{n} r_i x_i\right) &= f\left(\sum_{i=1}^{n} r_i x_i\right) - f(0) \\
&= \sum_{i=1}^{n} r_i f(x_i) - \sum_{i=1}^{n} r_i f(0) \\
&= \sum_{i=1}^{n} r_i \big(f(x_i) - f(0)\big) \\
&= \sum_{i=1}^{n} r_i g(x_i). \qquad (2.2)
\end{aligned}
$$

Thus g satisfies (NJ). Let $s \in \{1, \ldots n\}$,set $x_s = x$ and $x_1 = \ldots = x_{s-1} = x_{s+1} = \ldots = x_n = 0$, then (2.2) becomes

$$g(r_s x) = r_s g(x) \quad \text{for all} \quad s \in \{1, \ldots n\} \quad \text{for all} \quad x \in X. \qquad (2.3)$$

Next, by setting $x_s = x, x_{s+1} = y$ and $x_1 = \ldots = x_{s-1} = x_{s+2} = \ldots = x_n = 0$ in (2.2) and using (2.3), we obtain

$$g(r_s x + r_{s+1} y) = r_s g(x) + r_{s+1} g(y) \qquad (2.4)$$

for all $x, y \in X$. Let $u, v \in X$, set $x = r_s^{-1} u$ and $y = r_{s+1}^{-1} v$. From (2.3) and (2.4), we get

$$
\begin{aligned}
g(u + v) &= g(r_s x) + r_{s+1} y) \\
&= r_s g(x) + r_{s+1} g(y) \\
&= r_s g(r_s^{-1} u) + r_{s+1} g(r_{s+1}^{-1} v) \\
&= g(r_s r_s^{-1} u) + g(r_{s+1} r_{s+1}^{-1} v) \\
&= g(u) + g(v). \qquad (2.5)
\end{aligned}
$$

Therefore g is an additive function and, by definition of g, we get $f(x) = g(x) + f(0)$ for all $x \in X$.

(Sufficiency) Suppose $f(x) = A(x) + k$ for all $x \in X$ where $A : X \to Y$ is an additive function and k is a constant. Then,

$$\begin{aligned} f\left(\sum_{i=1}^{n} r_i x_i\right) &= A\left(\sum_{i=1}^{n} r_i x_i\right) - f(0) \\ &= A\left(\sum_{i=1}^{n} r_i x_i\right) - \sum_{i=1}^{n} r_i f(0) \\ &= \sum_{i=1}^{n} r_i (A(x_i) - f(0)) \\ &= \sum_{i=1}^{n} r_i f(x_i). \end{aligned}$$

□

The following theorem shows the generalized stability of (NJ). In investigating the stability of n-dimensional Jensen type functional equation we assume, in addition, that Y is a Banach space. For a function $\phi : X^n \to [0, \infty)$, we define for each $s = 1, \ldots, n$, a function $\phi_s : X \to [0, \infty)$ by

$$\phi_s(x) = \phi(\underbrace{0, \ldots, 0}_{s-1}, x, \underbrace{0, \ldots, 0}_{n-s}) \tag{2.6}$$

for all $x \in X$.

Theorem 2.2. *Assume that $\phi : X^n \to [0, \infty)$ and $f : X \to Y$ satisfy the following conditions*

(i) $\sum\limits_{i=0}^{\infty} r_s^{-i} \phi_s(r_s^i x)$ *converges,*

(ii) $\lim\limits_{m\to\infty} r_s^{-m} \phi(r_s^m x_1, \ldots, r_s^m x_n) = 0$ *for all* $x_1, \ldots, x_n \in X$,

(iii) $\left\| \sum_{i=1}^{n} r_i f(x_i) - f\left(\sum_{i=1}^{n} r_i x_i \right) \right\| \leq \phi(x_1, \ldots, x_n)$ *for all* $x_1, \ldots, x_n \in X$.

Then there exists a unique function $L : X \to Y$ that satisfies functional equation (NJ) and

$$\|f(x) - L(x)\| \leq \sum_{i=0}^{\infty} r_s^{-i-1} \phi_s(r_s^i x) \tag{2.7}$$

for all $x \in X$. Moreover, L is given by

$$L(x) = f(0) + \lim_{m\to\infty} r_s^{-m}\big(f(r_s^m x) - f(0)\big) \tag{2.8}$$

for all $x \in X$.

Proof. Define a function $g : X \to Y$ by

$$g(x) = f(x) - f(0) \tag{2.9}$$

for all $x \in X$. It should be noted that $g(0) = 0$. By (2.1),

$$\left\| \sum_{i=1}^{n} r_i g(x_i) - g\left(\sum_{i=1}^{n} r_i x_i \right) \right\| \leq \phi(x_1, \dots, x_n) \tag{2.10}$$

for all $x_1, \dots, x_n \in X$. Let $s \in \{1, \dots, n\}$. Set $x_s = x$ and $x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0$, then (2.10) becomes

$$\big\| r_s g(x) - g(r_s x) \big\| \leq \phi_s(x) \tag{2.11}$$

for all $x \in X$. Rewrite the above equation to

$$\big\| g(x) - r_s^{-1} g(r_s x) \big\| \leq r_s^{-1}\phi_s(x) \tag{2.12}$$

for all $x \in X$. For each positive integer m and each $x \in X$, we have

$$\begin{aligned}
\big\| g(x) - r_s^{-m} g(r_s^m x) \big\| &= \left\| \sum_{i=0}^{m-1} \Big(r_s^{-i} g(r_s^i x) - r_s^{-(i+1)} g(r_s^{i+1} x) \Big) \right\| \\
&\leq \sum_{i=0}^{m-1} \big\| r_s^{-i} g(r_s^i x) - r_s^{-(i+1)} g(r_s^{i+1} x) \big\| \\
&= \sum_{i=0}^{m-1} r_s^{-i} \big\| g(r_s^i x) - r_s^{-1} g(r_s r_s^i x) \big\| \\
&\leq \sum_{i=0}^{m-1} r_s^{-i-1} \phi_s(r_s^i x).
\end{aligned} \tag{2.13}$$

Consider the sequence $\{r_s^{-m} g(r_s^m x)\}$. For each positive integers $k < l$ and for each

$x \in X$,

$$\begin{aligned}\left\|r_s^{-k}g(r_s^k x) - r_s^{-l}g(r_s^l x)\right\| &= r_s^{-k}\left\|g(r_s^k x) - r_s^{-(l-k)}g(r_s^{l-k}r_s^k x)\right\| \\ &\le r_s^{-k}\sum_{i=0}^{l-k-1} r_s^{-i-1}\phi_s(r_s^{i+k}x) \\ &\le r_s^{-k-1}\sum_{i=0}^{\infty} r_s^{-i}\phi_s(r_s^{i+k}x).\end{aligned}$$

Since $\sum\limits_{i=0}^{\infty} r_s^{-i}\phi(r_s^i x)$ converges, $\lim\limits_{k\to\infty} r_s^{-k-1}\sum\limits_{i=0}^{\infty} r_s^{-i}\phi_s(r_s^{i+k}x) = 0$. This implies that

$$L(x) = f(0) + \lim_{m\to\infty} r_s^{-m}g(r_s^m x) \tag{2.14}$$

is well-defined in the Banach space Y. Moreover, as $m \to \infty$, (2.13) becomes

$$\left\|g(x) + f(0) - L(x)\right\| \le \sum_{i=0}^{\infty} r_s^{-i-1}\phi_s(r_s^i x).$$

Recalling the definition of $g(x)$, we see that inequality (2.7) is valid.

To show that L indeed satisfies (NJ), replace each x_i in (2.10) with $r_s^m x_i$,

$$\left\|\sum_{i=1}^{n} r_i g(r_s^m x_i) - g\left(r_s^m \sum_{i=1}^{n} r_i x_i\right)\right\| \le \phi(r_s^m x_1, \ldots, r_s^m x_n). \tag{2.15}$$

If we multiply the above inequality by r_s^{-m} and take the limit as $m \to \infty$, then by the definition of L in (2.14) and (2.1), we obtain

$$\left\|\sum_{i=1}^{n} r_i L(x_i) - L\left(\sum_{i=1}^{n} r_i x_i\right)\right\| \le \lim_{m\to\infty} r_s^{-m}\phi(r_s^m x_1, \ldots, r_s^m x_n) = 0, \tag{2.16}$$

which implies that

$$\sum_{i=1}^{n} r_i L(x_i) = L\left(\sum_{i=1}^{n} r_i x_i\right) \tag{2.17}$$

for all $x_1, \ldots, x_n \in X$.

To prove the uniqueness, suppose there is another function $L' : X \to Y$ satisfying (NJ) and (2.7). Observe that if we replace x_s by x and put $x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0$ in (2.17), then

$$r_s L(x) + (1 - r_s)L(0) = L(r_s x) \tag{2.18}$$

for all $x \in X$, and

$$L(0) = f(0) + \lim_{m\to\infty} r_s^{-m} g(0) = f(0).$$

The function $L^{'}$ obviously possesses the same properties. Therefore,

$$r_s\big(L(x) - L^{'}(x)\big) = L(r_s x) - L^{'}(r_s x) \tag{2.19}$$

for all $x \in X$. We can prove by mathematical induction that for each positive integer m,

$$r_s^m\big(L(x) - L^{'}(x)\big) = L(r_s^m x) - L^{'}(r_s^m x)$$

for all $x \in X$. Therefore, for each positive integer m,

$$\begin{aligned}\big\|L(x) - L^{'}(x)\big\| &= r_s^{-m}\big\|L(r^m x) - L^{'}(r_s^m x)\big\| \\ &\le r_s^{-m}\big(\big\|L(r^m x) - f(r_s^m x)\big\| + \big\|L^{'}(r_s^m x) - f(r_s^m x)\big\|\big) \\ &\le 2r_s^{-m}\sum_{i=0}^{\infty} r_s^{-i-1}\phi_s(r_s^{i+m}x)\end{aligned}$$

for all $x \in X$. Since $\sum\limits_{i=0}^{\infty} r_s^{-i}\phi(r_s^i x)$ converges, $\lim\limits_{m\to\infty} r_s^{-m}\sum\limits_{i=0}^{\infty} r_s^{-i-1}\phi(r_s^{i+m}x) = 0$. We conclude that $L(x) = L^{'}(x)$ for all $x \in X$. $\square$

Theorem 2.3. *Assume that $\phi : X^n \to [0,\infty)$ and $f : X \to Y$ satisfy the following conditions*

(i) $\sum\limits_{i=0}^{\infty} r_s^i\phi_s(r_s^{-i}x)$ *converges,*

(ii) $\lim\limits_{m\to\infty} r_s^m\phi(r_s^{-m}x_1, \dots, r_s^{-m}x_n) = 0$ *for all* $x_1, \dots, x_n \in X$,

(iii) $\left\| \sum_{i=1}^n r_i f(x_i) - f\left(\sum_{i=1}^n r_i x_i\right)\right\| \le \phi(x_1, \dots, x_n)$ *for all* $x_1, \dots, x_n \in X$.

Then there exists a unique function $L : X \to Y$ that satisfies functional equation (NJ) and

$$\big\|f(x) - L(x)\big\| \le \sum_{i=1}^{\infty} r_s^{i-1}\phi_s(r_s^{-i}x) \tag{2.20}$$

for all $x \in X$. Moreover, L is given by

$$L(x) = f(0) + \lim_{m\to\infty} r_s^m(f(r_s^{-m}x) - f(0)) \tag{2.21}$$

for all $x \in X$.

Proof. Referring the process (2.9)-(2.12), we can replace inequality (2.12) with

$$\left\|g(x) - r_s g(r_s^{-1}x)\right\| \leq \phi_s(r_s^{-1}x)$$

for all $x \in X$. For each positive integer m and each $x \in X$, we get

$$\begin{aligned}
\left\|g(x) - r_s^m g(r_s^{-m}x)\right\| &= \left\|\Big(\sum_{i=1}^{m} r_s^{i-1} g(r_s^{-(i-1)}x) - r_s^i g(r_s^{-i}x)\Big)\right\| \\
&\leq \sum_{i=1}^{m} \left\|r_s^{i-1} g(r_s^{-(i-1)}x) - r_s^i g(r_s^{-i}x)\right\| \\
&= \sum_{i=1}^{m} r_s^{i-1} \left\|g(r_s^{-(i-1)}x) - r_s g(r_s^{-1}r_s^{-(i-1)}x)\right\| \\
&\leq \sum_{i=1}^{m} r_s^{i-1}\phi_s(r_s^{-i}x).
\end{aligned} \tag{2.22}$$

We investigate the sequence $\{r_s^m g(r_s^{-m}x)\}$. For each positive integer $k < l$ and each $x \in X$,

$$\begin{aligned}
\left\|r_s^k g(r_s^{-k}x) - r_s^l g(r_s^{-l}x)\right\| &= r_s^k \left\|g(r_s^{-k}x) - r_s^{l-k} g(r_s^{-(l-k)} r_s^{-k}x)\right\| \\
&\leq r_s^k \sum_{i=1}^{l-k} r_s^{i-1}\phi_s(r_s^{-i-k}x) \\
&\leq r_s^{k-1} \sum_{i=1}^{\infty} r_s^{i}\phi_s(r_s^{-i-k}x).
\end{aligned}$$

Since $\sum_{i=0}^{\infty} r_s^i \phi(r_s^{-i}x)$ converges, $\lim_{k\to\infty} r_s^{k-1} \sum_{i=0}^{\infty} r_s^i \phi_s(r_s^{-i-k}x) = 0$. Thus,

$$L(x) = f(0) + \lim_{m\to\infty} r_s^m g(r_s^{-m}x) \tag{2.23}$$

is well-defined in the Banach space Y. Furthermore, (2.22) becomes as $m \to \infty$,

$$\left\|g(x) + f(0) - L(x)\right\| \leq \sum_{i=1}^{\infty} r_s^{i-1}\phi_s(r_s^{-i}x).$$

By the definition of $g(x)$, inequality (2.20) is valid.

In order to show that L satisfies (NJ). We replace each x_i by $r_s^{-m}x_i$ in (2.10) and multiply r_s^m, then take the limit as $m \to \infty$, we have

$$\left\| \sum_{i=1}^{n} r_i L(x_i) - L\left(\sum_{i=1}^{n} r_i x_i \right) \right\| \leq \lim_{m\to\infty} r_s^m \phi(r_s^{-m}x_1, \ldots, r_s^{-m}x_n) = 0,$$

which implies (2.17).

To prove the uniqueness, suppose there is another function $L' : X \longrightarrow Y$ satisfying (NJ) and (2.7). Replacing x by $r_s^{-1}x$ and put $x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0$ in (2.17); consequently, (2.19) becomes

$$r_s\big(L(r_s^{-1}x) - L^{'}(r_s^{-1}x)\big) = L(x) - L^{'}(x).$$

For each positive m, we can show by mathematical induction that

$$r_s^m\big(L(r_s^{-m}x) - L^{'}(r_s^{-m}x)\big) = L(x) - L^{'}(x)$$

for all $x \in X$. Therefore, for each positive integer m,

$$\begin{aligned} \big\|L(x) - L^{'}(x)\big\| &= r_s^m \big\|L(p^{-m}x) - L^{'}(r_s^{-m}x)\big\| \\ &\leq r_s^m\big(\big\|L(r^{-m}x) - f(r_s^{-m}x)\big\| + \big\|L^{'}(r_s^{-m}x) - f(r_s^{-m}x)\big\|\big) \\ &\leq 2r_s^m \sum_{i=1}^{\infty} r_s^{i-1}\phi_s(r_s^{-i-m}x) \end{aligned}$$

for all $x \in X$. Since $\sum\limits_{i=1}^{\infty} r_s^i \phi(r_s^{-i}x)$ converges, $\lim\limits_{m\to\infty} r_s^m \sum\limits_{i=0}^{\infty} r_s^{i-1}\phi(r_s^{-i-m}x) = 0$. We obtain that $L(x) = L^{'}(x)$ for all $x \in X$. $\square$

Theorem 2.4. *Let $\varepsilon > 0$ be a real number. If a function $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{n} r_i f(x_i) - f\left(\sum_{i=1}^{n} r_i x_i \right) \right\| \leq \varepsilon \tag{2.24}$$

for all $x_1, \ldots, x_n \in X$, then there exists a unique function $L : X \to Y$ that satisfies (NJ) and

$$\big\|f(x) - L(x)\big\| \leq \frac{\varepsilon}{1 - r_{\min}}$$

for all $x \in X$, where $r_{\min} = \min\{r_1, \ldots, r_n\}$.

Proof. Let

$$\phi(x_1, \dots, x_n) = \varepsilon$$

for all $x_1, \dots, x_n \in X$ in Theorem 2.3. We can see that Theorem 2.3 holds for every $s = 1, \dots, n$. We choose s such that $r_s = r_{\min} = \min\{r_1, \dots, r_n\}$. Then (2.20) becomes

$$\|f(x) - L(x)\| \leq \varepsilon \sum_{i=1}^{\infty} r_s^{i-1} = \frac{\varepsilon}{1 - r_s} = \frac{\varepsilon}{1 - r_{\min}}$$

for all $x \in X$ as desired. □

The following theorem proves the stability of (NJ).

Theorem 2.5. *Let $\varepsilon > 0$ and $m > 0$ be real numbers with $m \neq 1$. If a function $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{n} r_i f(x_i) - f\left(\sum_{i=1}^{n} r_i x_i \right) \right\| \leq \varepsilon \sum_{i=1}^{n} \|x_i\|^m \tag{2.25}$$

for all $x_1, \dots, x_n \in X$, then there exists a unique function $L : X \to Y$ that satisfies (NJ) and

$$\|f(x) - L(x)\| \leq \frac{\varepsilon}{M} \|x\|^m$$

for all $x \in X$, where $M = \max_{i=1,\dots,n} \mid r_i - r_i^m \mid$.

Proof. In the case $0 < m < 1$, let

$$\phi(x_1, \dots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^m$$

for all $x_1, \dots, x_n \in X$ in Theorem 2.3. Then we can see that Theorem 2.3 holds for every $s = 1, \dots, n$. We choose s such that

$$\mid r_s - r_s^m \mid = M = \max_{i=1,\dots,n} \mid r_i - r_i^m \mid .$$

Thus, (2.7) becomes

$$\begin{aligned} \|f(x) - L(x)\| &\leq \varepsilon \sum_{i=1}^{\infty} r_s^{i-1} \|r_s^{-i} x\|^m = \varepsilon \|x\|^m r_s^{-1} \sum_{i=1}^{\infty} r_s^{i(1-m)} \\ &= \varepsilon r_s^{-1} \|x\|^m \left(\frac{r_s^{1-m}}{1 - r_s^{1-m}} \right) = \frac{\varepsilon}{r_s^m - r_s} \|x\|^m = \frac{\varepsilon}{M} \|x\|^m \end{aligned}$$

for all $x \in X$. In the case $m > 1$, let

$$\phi(x_1, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^m$$

for all $x_1, \ldots, x_n \in X$ in Theorem 2.1. Since Theorem 2.1 holds for every $s = 1, \ldots, n$, (2.7) becomes

$$\begin{aligned}\big\|f(x) - L(x)\big\| &\le \varepsilon \sum_{i=0}^{\infty} r_s^{-i-1} \|r_s^i x\|^m = \varepsilon \|x\|^m r_s^{-1} \sum_{i=0}^{\infty} r_s^{i(m-1)} \\ &= \varepsilon \|x\|^m r_s^{-1} \Big(\frac{1}{1 - r_s^{m-1}} \Big) = \frac{\varepsilon}{r_s - r_s^m} \|x\|^m = \frac{\varepsilon}{M} \|x\|^m\end{aligned}$$

for all $x \in X$. $\square$

CHAPTER III
ALTERNATIVE QUADRATIC FUNCTIONAL EQUATION ON 2-DIVISIBLE ABELIAN GROUPS

Throughout this chapter, let f be a function from a 2-divisible abelian group to a real (or rational or complex) linear space Y. We will prove that the alternative quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)\pm 2f(y), \tag{AQ}$$

is equivalent to the quadratic functional equation (Q) for functions $f : G \to Y$. It should be noted that the 2-divisibility property of the domain of the function has been extensively used in the work of Skof [15], but here we will present a proof which relies on a minimal use of 2-divisibility.

Many substitutions will be made through the alternative quadratic functional equation (AQ), it therefore is quite convenient to adopt the following notations for the rest of this chapter:

$$\begin{aligned}\mathcal{AQ}_f(x,y) := \Big(&f(x+y)+f(x-y)=2f(x)+2f(y)\\ &\text{or}\quad f(x+y)+f(x-y)=2f(x)-2f(y)\Big).\end{aligned}$$

First of all, we will give two of the most basic properties that can be derived directly from the alternative quadratic functional equation (AQ).

Lemma 3.1. *If a function $f : G \to Y$ satisfies (AQ) for all $x, y \in G$, then*

1. *$f(0) = 0$, and*

2. *$f(-x) = f(x)$ for all $x \in G$, i.e. f is an even function.*

Proof. Suppose that $f : G \to Y$ satisfies (AQ) for all $x, y \in G$.

1. Considering $\mathcal{AQ}_f(0,0)$, we immediately get $f(0) = 0$.

2. Assume that there exists $x_0 \in G$ such that

$$f(-x_0) \neq f(x_0) \tag{3.1}$$

   Considering $\mathcal{AQ}_f(0,x_0)$ and substituting $f(0) = 0$, we can see that

$$f(-x_0) = f(x_0) \quad \text{or} \quad f(-x_0) = -3f(x_0). \tag{3.2}$$

   Similarly, $\mathcal{AQ}_f(0,-x_0)$ with $f(0) = 0$ gives

$$f(-x_0) = f(x_0) \quad \text{or} \quad f(x_0) = -3f(-x_0). \tag{3.3}$$

   Taking the assumption (3.1) into account, we can derive from (3.2) and (3.3) that

$$f(-x_0) = -3f(x_0) \quad \text{and} \quad f(x_0) = -3f(-x_0)..$$

   Solving the above equation gives

$$f(-x_0) = 0 \quad \text{and} \quad f(x_0) = 0,$$

   which contradicts the assumption (3.1).
   Therefore, $f(-x) = f(x)$ for all $x \in G$ as desired.

□

The next lemma will use the 2-divisibility property of the group G to prove that $f(2x) = 4f(x)$ for all $x \in G$, where f is any function satisfying (AQ).

Lemma 3.2. *If a function $f : G \to Y$ satisfies (AQ) for all $x, y \in G$, then $f(2x) = 4f(x)$ for all $x \in G$.*

Proof. Suppose that $f : G \to Y$ satisfies (AQ) for all $x, y \in G$.
For any $x \in G$, $\mathcal{AQ}_f(x,x)$ with $f(0) = 0$ simplifies to

$$f(2x) = 4f(x) \quad \text{or} \quad f(2x) = 0 \quad \text{for all } x \in G. \tag{3.4}$$

Assume that there exists $x_0 \in G$ such that $f(2x_0) \neq 4f(x_0)$. Since G is a 2-divisible group, we let $z_0 \in G$ such that $2z_0 = x_0$. Therefore,

$$f(4z_0) \neq 4f(2z_0). \tag{3.5}$$

With $x = 2z_0$ in (3.4) and taking the assumption (3.5) into account, we have

$$f(4z_0) = 0. \tag{3.6}$$

From (3.5) and (3.6), we know that

$$f(2z_0) \neq 0 \tag{3.7}$$

With $x = z_0$ in (3.4) and taking (3.7) into account, we now have

$$f(2z_0) = 4f(z_0). \tag{3.8}$$

$\mathcal{AQ}_f(2z_0, z_0)$ with $f(2z_0)$ from (3.8) gives

$$f(3z_0) = 9f(z_0) \quad \text{or} \quad f(3z_0) = 5f(z_0). \tag{3.9}$$

$\mathcal{AQ}_f(3z_0, z_0)$ with $f(4z_0)$ from (3.6) and $f(2z_0)$ from (3.8) gives

$$f(3z_0) = f(z_0) \quad \text{or} \quad f(3z_0) = 3f(z_0). \tag{3.10}$$

Considering all possibilities in (3.9) and (3.10), we infer that $f(z_0) = 0$, which contradicts (3.7) and (3.8).

Therefore, $f(2x) = 4f(x)$ for all $x \in G$ as desired. $\square$

The following lemma will now generalize Lemma 3.1 and Lemma 3.2 to the following important lemma which actually proves that f is integrally homogeneous of degree 2.

Lemma 3.3. *If a function $f : G \to Y$ satisfies (AQ) for all $x, y \in G$, then*

$$f(nx) = n^2 f(x) \quad \textit{for all } x \in G \textit{ and for all } n \in \mathbb{Z}.$$

Proof. Suppose that $f : G \to Y$ satisfies (AQ) for all $x, y \in G$, and let $x \in G$. We will first prove by mathematical induction that, for all $n = -1, 0, 1, \ldots$,

$$f(nx) = n^2 f(x). \tag{3.11}$$

For $n = 1$, (3.11) is trivial. While for $n = -1, 0, 2$, (3.11) follows from Lemma 3.1 and Lemma 3.2.

Now suppose that $f(kx) = k^2 f(x)$ for all $k = -1, 0, 1, \ldots, n$ for an integer $n \geq 2$. We will prove that $f\big((n+1)x\big) = (n+1)^2 f(x)$ by a contradiction; i.e., assume that

$$f\big((n+1)x\big) \neq (n+1)^2 f(x). \tag{3.12}$$

$\mathcal{AQ}_f(nx, x)$ with $f\big((n-1)x\big)$ and $f(nx)$ from the induction hypothesis will simplify to

$$f\big((n+1)x\big) = (n+1)^2 f(x) \quad \text{or} \quad f\big((n+1)x\big) = \big(n^2 + 2n - 3\big)\, f(x).$$

Taking the assumption (3.12) into account, we are left with

$$f\big((n+1)x\big) = \big(n^2 + 2n - 3\big)\, f(x). \tag{3.13}$$

$\mathcal{AQ}_f\big((n-1)x, 2x\big)$ with $f\big((n-3)x\big), f\big((n-1)x\big)$ and $f(2x)$ from the induction hypothesis, will simplify to

$$f\big((n+1)x\big) = (n+1)^2 f(x) \quad \text{or} \quad f\big((n+1)x\big) = \big(n^2 + 2n - 15\big)\, f(x).$$

Taking the assumption (3.12) into account, we are left with

$$f\big((n+1)x\big) = (n^2 + 2n - 15) f(x). \tag{3.14}$$

Equating (3.13) and (3.14) will lead to the conclusion that

$$f(x) = 0 \quad \text{and} \quad f\big((n+1)x\big) = 0,$$

which contradicts (3.12).

Therefore, (3.11) holds for all $n = -1, 0, 1, \ldots$.

Lemma 3.1 tells us that f is an even function; therefore, (3.11) also holds for all negative integers n. This completes the proof of the lemma. $\square$

The following theorem will show that the alternative quadratic functional equation (AQ) is equivalent to the quadratic functional equation.

Theorem 3.4. *A function $f : G \to Y$ satisfies (AQ) for all $x, y \in G$ if and only if f satisfies (Q) for all $x, y \in G$.*

Proof. ($\Leftarrow$) A function $f : G \to Y$ satisfies (Q) for all $x, y \in G$, then it is obvious that f satisfies (AQ) for all $x, y \in G$.

($\Rightarrow$) Suppose that a function $f : G \to Y$ satisfies (AQ) for all $x, y \in G$.
We will prove that f satisfies (Q) for all $x, y \in G$ by a contradiction.
Suppose there exist $x_0, y_0 \in G$ such that

$$f(x_0 + y_0) + f(x_0 - y_0) \neq 2f(x_0) + 2f(y_0). \tag{3.15}$$

The assumption (3.15) will be used to eliminate an alternative from $\mathcal{AQ}_f(x, y)$ for many suitable choices of x and y.

In order to better understand the ideas, we will divide the proof into a few steps.

Step 1: Determine $f(x_0 + y_0), f(x_0 - y_0), f(y_0)$ in terms of $f(x_0)$.
$\mathcal{AQ}_f(x_0, y_0)$ with (3.15) gives

$$f(x_0 + y_0) + f(x_0 - y_0) = 2f(x_0) - 2f(y_0). \tag{3.16}$$

$\mathcal{AQ}_f(x_0 + y_0, x_0 - y_0)$ with $f(2x) = 4f(x)$ simplifies to

$$\begin{aligned} & 2f(x_0) + 2f(y_0) = f(x_0 + y_0) + f(x_0 - y_0) \\ \text{or} \quad & 2f(x_0) + 2f(y_0) = f(x_0 + y_0) - f(x_0 - y_0). \end{aligned}$$

Taking the assumption (3.15) into account, we are left with

$$f(x_0 + y_0) - f(x_0 - y_0) = 2f(x_0) + 2f(y_0). \tag{3.17}$$

$\mathcal{AQ}_f(y_0, x_0)$ with $f(-x) = f(x)$ as well as the assumption (3.15) gives

$$f(x_0 + y_0) + f(x_0 - y_0) = 2f(y_0) - 2f(x_0). \tag{3.18}$$

For convenience, we will let

$$a := f(x_0).$$

Equating (3.16) and (3.18) yields

$$f(y_0) = a.$$

From (3.16) and (3.17), we get that

$$f(x_0 + y_0) = 2a \quad \text{and} \quad f(x_0 - y_0) = -2a.$$

The values of $f(x_0 + y_0), f(x_0 - y_0), f(y_0)$ and $f(x_0)$ in terms of a will be regarded as known and will be used in the subsequent steps.

Step 2: Determine all possible values of $f(x_0 + 2y_0)$ and $f(x_0 - 2y_0)$.
$\mathcal{AQ}_f(x_0 + 2y_0, x_0 - 2y_0)$ with $f(2x) = 4f(x)$ and $f(4x) = 16f(x)$ gives

$$10a = f(x_0 + 2y_0) \pm f(x_0 - 2y_0). \tag{3.19}$$

$\mathcal{AQ}_f(x_0 - y_0, -y_0)$ with $f(-x) = f(x)$ gives

$$f(x_0 - 2y_0) \in \{-3a, -7a\}. \tag{3.20}$$

Substituting $f(x_0 - 2y_0)$ from (3.19) into (3.20) gives 4 possible values for $f(x_0 + 2y_0)$:

$$f(x_0 + 2y_0) \in \{3a, 7a, 13a, 17a\}. \tag{3.21}$$

$\mathcal{AQ}_f(x_0 + y_0, y_0)$ gives 2 possible values for $f(x_0 + 2y_0)$:

$$f(x_0 + 2y_0) \in \{a, 5a\}. \tag{3.22}$$

Step 3: Put the jigsaw together to conclude the value of a.
From the values of $f(x_0 + 2y_0)$ in (3.21) and (3.22), we can conclude that $a = 0$, which in turn implies that

$$f(x_0 + y_0) = f(x_0 - y_0) = f(x_0) = f(y_0) = 0$$

and eventually contradict the assumption (3.15).

Therefore, f satisfies (Q) as desired. □

The following example will show that the 2-divisibility of G is crucial to the equivalence of (AQ) and (Q). It should be emphasize that the 2-divisibility has been used only in the proof of Lemma. 3.2.

Example 3.5. *Consider a function $f : \mathbb{Z} \to \mathbb{R}$ defined by*

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

for all $n \in \mathbb{Z}$.

For any $m, n \in \mathbb{Z}$, we can see that $m+n$ and $m-n$ are always of the same parity. Therefore,

$$f(m+n) + f(m-n) = \begin{cases} 0 & \text{if} \quad m \equiv n \pmod 2, \\ 2 & \text{if} \quad m \not\equiv n \pmod 2. \end{cases}$$

Moreover, if $m \equiv n \pmod 2$, then $f(m) - f(n) = 0$,
and if $m \not\equiv n \pmod 2$, then $f(m) + f(n) = 1$. Hence,

$$f(m+n) + f(m-n) = 2f(m) \pm 2f(n);$$

that is, f satisfies (AQ) for all $m, n \in \mathbb{Z}$.

However, one can easily verify that

$$f(2) + f(0) \neq 2f(1) + 2f(1).$$

Hence, f does not satisfy (Q).

Therefore, the alternative quadratic functional equation (AQ) is not equivalent to the quadratic functional equation (Q) for this particular function f defined on the group $(\mathbb{Z}, +)$ which is not 2-divisible.

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