สมการเชิงฟังก์ชันเจนเซนทางเลือกบนกรุป

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วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์และวิทยาการคณนา ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2558

ALTERNATIVE JENSEN FUNCTIONAL EQUATION ON GROUPS

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy Program in Applied Mathematics and Computational Science

Department of Mathematics and Computer Science

Faculty of Science

Chulalongkorn University

Academic Year 2015

Thesis Title	ALTERNATIVE JENSEN FUNCTIONAL EQUATION ON GROUPS
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ชูเดช ศรีสวัสดิ์ : สมการเชิงฟังก์ชันเจนเซนทางเลือกบนกรุป (ALTERNATIVE JENSEN FUNCTIONAL EQUATION ON GROUPS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : รศ.ดร. ไพศาล นาคมหาชลาสินธุ์, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม : ผศ.ดร. ณัฐพันธ์ กิติสิน, 55 หน้า.

กำหนดจำนวนเต็ม α, β, γ ซึ่ง $(\alpha, \beta, \gamma) \neq k(1, -2, 1)$ สำหรับทุก $k \in \mathbb{Z}$ เราจะสร้างเกณฑ์สำหรับความมีอยู่ของผลเฉลยทั่วไปของสมการเชิงฟังก์ชันเจนเซนทางเลือกซึ่งอยู่ในรูปแบบ

$$f(xy^{-1}) - 2f(x) + f(xy) = 0 \quad \text{หรือ} \quad \alpha f(xy^{-1}) + \beta f(x) + \gamma f(xy) = 0$$

เมื่อ f คือฟังก์ชันจากกรุป $(G, \cdot)$ ไปกรุปสลับที่ที่หารได้เพียงตัวเดียว $(H, +)$

ภาควิชา : คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต : . . . .
สาขาวิชา : คณิตศาสตร์ประยุกต์และวิทยาการคณนา	ลายมือชื่อ อ.ที่ปรึกษาหลัก : . . . .
ปีการศึกษา : 2558	ลายมือชื่อ อ.ที่ปรึกษาร่วม : . . . .

5472819323 : MAJOR APPLIED MATHEMATICS AND COMPUTATIONAL SCIENCE

KEYWORDS : JENSEN EQUATION/ ALTERNATIVE FUNCTIONAL EQUATION/ ALTERNATIVE JENSEN

CHOODECH SRISAWAT : ALTERNATIVE JENSEN FUNCTIONAL EQUATION ON GROUPS. ADVISOR : ASSOC. PROF. PAISAN NAKMAHACHALASINT, Ph.D., CO-ADVISOR : ASST. PROF. NATAPHAN KITISIN, Ph.D., 55 pp.

Given integers α, β, γ such that $(\alpha, \beta, \gamma) \neq k(1, -2, 1)$ for all $k \in \mathbb{Z}$, we will establish a criterion for the existence of the general solution of the alternative Jensen functional equation of the form

$$f(xy^{-1}) - 2f(x) + f(xy) = 0 \quad \text{or} \quad \alpha f(xy^{-1}) + \beta f(x) + \gamma f(xy) = 0,$$

where f is a mapping from a group $(G, \cdot)$ to a uniquely divisible abelian group $(H, +)$.

Department : Mathematics and Computer Science .... Student's Signature : ....

Field of Study : Applied Mathematics and Computational Science Advisor's Signature : ....

Academic Year : 2015 .... Co-advisor's Signature : ....

ACKNOWLEDGEMENTS

In the completion of the thesis, I am deeply indebted to my advisor, Associate Professor Dr. Paisan Nakmahachalasint and my co-advisor, Assistant Professor Dr. Nataphan Kitisin, not only for their guidance to research but also for broadening my academic vision. They also constantly encouraged me to overcome any obstacles I encountered during this work. Meanwhile, I could not have finished this thesis without their inspiration and insightful instruction.

I would like to extend my gratitude to all dissertation committee: Assistant Professor Dr. Khamron Mekchay, Associate Professor Dr. Patanee Udomkavanich, Assistant Professor Dr. Keng Wiboonton and Dr. Tippaporn Eungrasamee for their comments and helpful suggestions. Furthermore, I would like to thank all teachers who have taught me my valuable knowledge.

Next, I would like to express my gratitude to Chulalongkorn University and Science Achievement Scholarship of Thailand, the scholarship grantor entity for the support throughout my study. Without this financial support, my research could not go on.

Last but not least, I am most grateful to my family, the most important thing in my life, for their understanding and whose patient love enabled me to complete this thesis. I really owe what I am to them; especially, my mother and my father who make me believe in my own way and let me grow up as a quality man. My younger brother who looks at me as a role model also intensifies my effort in this graduation.

CONTENTS

CHAPTER I

INTRODUCTION

In this chapter, we will state some background in functional equations, Cauchy and Jensen Functional equation, alternative functional equations, our proposed problem and the notations used throughout this thesis.

1.1 Functional Equations

A functional equation is simply an equation of unknown functions. In order to solve the functional equation, we seek all possible functions satisfying the given functional equation.

Example 1.1.1. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the functional equation

$$f(x+y) = x + f(y), \qquad \text{for all } x, y \in \mathbb{Z}. \tag{1.1}$$

Solution. For each $x \in \mathbb{Z}$, by (1.1), we observe that

$$f(x+0) = x + f(0).$$

Hence there exists $c \in \mathbb{Z}$ namely $c = f(0)$ such that

$$f(x) = x + c \text{ for all } x \in \mathbb{Z}. \tag{1.2}$$

On the contrary, if f is given by (1.2), then f actually satisfies (1.1). Therefore, the general solution of (1.1) is given by (1.2).

However, some functional equations have no solution as the following example.

Example 1.1.2. Given a functional equation

$$f(x+y)+xf(x)=1 \qquad \text{for all } x,y\in\mathbb{Z}. \tag{1.3}$$

Solution. For each $y\in\mathbb{Z}$, we substitute $x=0$ in (1.3) to obtain $f(y)=1$, i.e., we have

$$f(1)=f(2)=1. \tag{1.4}$$

By (1.4), setting $x=y=1$ in (1.3), we get $2=1$, a contradiction. Therefore, there is no a function $f:\mathbb{Z}\to\mathbb{Z}$ satisfying (1.3).

1.2 Cauchy and Jensen Functional Equation

The additive functional equation,

$$f(x+y)=f(x)+f(y), \tag{1.5}$$

is one of the most well-known functional equations. In 1821, Cauchy [3] proved that all continuous solutions of (1.5) on $\mathbb{R}$ are given by $f(x)=cx$ for all $x\in\mathbb{R}$, where c is a constant in $\mathbb{R}$. Later on, the additive functional equation (1.5) was known as the *Cauchy functional equation*. In 1905, Hamel [7] constructed the general solution of (1.5) using a Hamel basis over $\mathbb{Q}$. Afterwards, Hewitt and Zukerman [8] gave a remarkable result of the nonlinear additive functions. In fact, the graph $G(f)=\{(x,f(x)) : x\in\mathbb{R}\}$ is a dense subset of $\mathbb{R}^2$, that is, given $\varepsilon>0$ and $(x,y)\in\mathbb{R}^2$, there exists $(a,f(a))\in G(f)$ such that $(x-a)^2+(y-f(a))^2<\varepsilon^2$, which indicates that the graph $G(f)$ of a nonlinear additive functions consists of points that disperse all over $\mathbb{R}^2$.

The *Jensen functional equation* is the equation of the form,

$$f\Big(\frac{x+y}{2}\Big)=\frac{f(x)+f(y)}{2}, \tag{1.6}$$

which is closely related to the Cauchy functional equation (1.5). In [4], it is shown that the general solution of (1.6) is of the form $f(x) = A(x) + c$, where A is an additive function, a solution of the Cauchy functional equation (1.5).

If the domain of the Jensen functional equation (1.6) is a group $(G, \cdot)$, then (1.6) can be written in the form

$$f(xy^{-1}) - 2f(x) + f(xy) = 0. \tag{J}$$

Many extensive works on Jensen functional equation on different kinds of groups have been widely studied (e.g., Le and Thai [12], Ng [14], [15], Parnami and Vasudeva [16], and Stetkær [19]).

1.3 Alternative Functional Equations

An alternative functional equation is a challenging problem in functional equations. Normally, a functional equation is a single equation with a function as a variable such as the Cauchy functional equation (1.5). However, in an alternative functional equation, there are more than one equation that the function has to satisfy. For instance, Kannappan and Kuczma [10] solved the alternative Cauchy functional equation

$$\left(f(x+y) - af(x) - bf(y)\right)\left(f(x+y) - f(x) - f(y)\right) = 0 \tag{1.7}$$

on an abelian group. This implies that the solution f has to satisfy

$$\left(f(x+y) - af(x) - bf(y)\right) = 0 \quad \text{or} \quad \left(f(x+y) - f(x) - f(y)\right) = 0.$$

Afterwards, Forti [5] has successfully found the general solution of (1.7) in a more general setting of the form

$$\left(cf(x+y) - af(x) - bf(y) - d\right)\left(f(x+y) - f(x) - f(y)\right) = 0.$$

Inspired by the work on the alternative Cauchy functional equation, Nakmahachalasint [13] first investigated the alternative Jensen functional equation of the form

$$f(x) \pm 2f(xy) + f(xy^2) = 0$$

on a semigroup.

Next, we will give an example of the general solution of the alternative Cauchy functional equation on $\mathbb{Z}$ as follows:

Example 1.3.1. Find the general solution $f : \mathbb{Z} \to \mathbb{Z}$ of the alternative Cauchy functional equation

$$f(x+y) = \pm(f(x) + f(y)) \quad \text{for all } x, y \in \mathbb{Z}. \tag{1.8}$$

Solution. In this example, we will prove that the general solution of (1.8) is exactly the solution of the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{Z}. \tag{1.9}$$

First, substituting $x = y = 0$ in (1.8), we have $f(0) = \pm 2f(0)$ and so

$$f(0) = 0. \tag{1.10}$$

For each $x \in \mathbb{Z}$, replacing y by $-x$ in (1.8), we get

$$f(0) = \pm(f(x) + f(-x)).$$

By (1.10), we conclude that

$$f(-x) = -f(x) \quad \text{for all } x \in \mathbb{Z}. \tag{1.11}$$

Suppose in a contrary that there exist $m, n \in \mathbb{Z}$ such that

$$f(m+n) \neq f(m) + f(n). \tag{1.12}$$

By (1.8), we obtain that

$$f(m+n) = -f(m) - f(n). \tag{1.13}$$

By (1.8), (1.11) and (1.13), we get

$$\begin{aligned} f(m) &= f(m+n-n) \\ &= \pm(f(m+n) + f(-n)) \\ &= \pm(-f(m) - f(n) - f(n)) \\ &= \pm(-f(m) - 2f(n)), \end{aligned}$$

which implies that

$$f(m) + f(n) = 0 \text{ or } f(n) = 0. \tag{1.14}$$

On the other hand, by (1.8), (1.11) and (1.13), we have

$$\begin{aligned} f(n) &= f(m+n-m) \\ &= \pm(f(m+n) + f(-m)) \\ &= \pm(-f(m) - f(n) - f(m)) \\ &= \pm(-2f(m) - f(n)), \end{aligned}$$

which gives

$$f(m) + f(n) = 0 \text{ or } f(m) = 0. \tag{1.15}$$

Combining (1.14) and (1.15), we get

$$f(m) + f(n) = 0 \text{ or } f(m) = f(n) = 0. \tag{1.16}$$

By (1.13) and (1.16), we have

$$f(m+n)=0. \tag{1.17}$$

From (1.16) and (1.17), we obtain that $f(m+n) = f(m)+f(n)$, a contradiction to (1.12). Therefore, we conclude that (1.9) must hold.

Conversely, if f is given by (1.9), then f also satisfies (1.8).

1.4 Proposed Problem

Motivated by the work of Nakmahachalasint [13] and Forti [5], we studied the alternative Jensen functional equation in a more general setting. In other words, given integers α, β, γ with

$$(\alpha,\beta,\gamma) \neq k(1,-2,1) \text{ for all } k \in \mathbb{Z}, \tag{1.18}$$

we will find a criterion of the existence of the general solution of the alternative Jensen functional equation of the form

$$f(xy^{-1})-2f(x)+f(xy)=0 \quad \text{or} \quad \alpha f(xy^{-1})+\beta f(x)+\gamma f(xy)=0, \tag{SA}$$

where f is a mapping from a group $(G,\cdot)$ to a uniquely divisible abelian group $(H,+)$. Then we show that, if $\beta = \alpha+\gamma$ or $(\beta,\gamma) \in \{(0,\alpha),(\alpha,\alpha)\}$, then the above alternative functional equation is equivalent to the Jensen functional equation (J), in the sense that their sets of solution are the same. Furthermore, we also find the general solution in the case when the domain G is a cyclic group.

1.5 Notations

Throughout the dissertation, we will use the following notations. Let $(G,\cdot)$ be a group and $(H,+)$ be a uniquely divisible abelian group. Next, we will introduce the notations for sequences $(a_k)_{k\in\mathbb{Z}}$ in H as follows.

Notation 1. We denote $(a_k)_{k\in\mathbb{Z}} = (\overline{\alpha},\overline{\beta})$ when there exists $k_0 \in \mathbb{Z}$ with $a_i = \alpha$ for all $i < k_0$

and $a_i = \beta$ for all $i \geq k_0$, i.e.,

$$(\ldots, \alpha, \alpha,\ \beta, \beta, \ldots) = (\overline{\alpha}, \overline{\beta}).$$

Notation 2. We denote $(a_k)_{k\in\mathbb{Z}} = (\overline{\alpha}, \beta, \overline{\gamma})$ when there exists $k_0 \in \mathbb{Z}$ with $a_i = \alpha$ for all $i < k_0$, $a_{k_0} = \beta$, and $a_i = \gamma$ for all $i > k_0$, i.e.,

$$(\ldots, \alpha, \alpha,\ \beta,\ \gamma, \gamma, \ldots) = (\overline{\alpha}, \beta, \overline{\gamma}).$$

Notation 3. Let p be a positive integer. We denote $(a_k)_{k\in\mathbb{Z}} = (\overline{\alpha_0, \ldots, \alpha_{p-1}})$ when there exists $k_0 \in \mathbb{Z}$ such that $a_i = \alpha_{k_0+i \pmod p}$ for all $i \in \mathbb{Z}$. In other words, $(\overline{\alpha_0, \ldots, \alpha_{p-1}})$ is a periodic sequence of a period p, i.e.,

$$(\ldots,\ \alpha_0, \ldots, \alpha_{p-1},\ \alpha_0, \ldots, \alpha_{p-1},\ \ldots) = (\overline{\alpha_0, \ldots, \alpha_{p-1}}).$$

CHAPTER II

THE n-DIMENSIONAL FUNCTIONAL EQUATION OF JENSEN TYPE

First, we will mention the question about the stability of functional equations. This problem was first introduced by Ulam [21] during his talk in the Mathematics Club of the University of Wisconsin. He proposed the question as follows:

"Let G_1 be a group and let G_2 be a metric group with the metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(f(x), H(x)) < \varepsilon$ for all $x \in G_1$?"

In the following year, Hyers [9] was the first to answer this stability problem. He found that for Banach spaces E_1 and E_2, if a mapping $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon,$$

for all $x, y \in E_1$ and for some $\varepsilon > 0$, then there exists a unique additive mapping $A : E_1 \to E_2$ satisfying the inequality

$$\|f(x) - A(x)\| \leq \varepsilon$$

and A takes the from of $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$. Afterwards, this notion was called as the *Hyers-Ulam stability* and leaded to one of fundamental concepts of the stability theory on functional equations. Aoki [1] and Bourgin [2] generalized Hyers' theorem for additive mappings by considering the bounded Cauchy differences. In 1978, Rassias [17] showed that if a mapping $f : E_1 \to E_2$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p),$$

for all $x, y \in E_1$ and for some $\theta \geq 0$ and $0 \leq p < 1$, then there exists a unique additive mapping

$A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p.$$

Later on, this type of stability is called the *Hyers-Ulam-Rassias stability*.

In this chapter, we will study the general solution and the generalized stability of the Jensen functional equation in more general setting of the form

$$\sum_{i=1}^{n} p_i f(x_i) = f\left(\sum_{i=1}^{n} p_i x_i\right), \tag{pJ}$$

where $n > 1$ is an integer, and $p_1, \ldots, p_n$ are positive rational numbers with

$$\sum_{i=1}^{n} p_i = 1. \tag{2.1}$$

Note that our work in this chapter appears in [20].

2.1 General Solution on an n-Dimensional Functional Equation of Jensen Type

In this section, we will give the general solution of (pJ) as the following theorem.

Theorem 2.1.1. *Let X and Y be real vector spaces. A mapping $f : X \to Y$ satisfies the functional equation (pJ) where $n > 1$ is an integer, and $p_1, \ldots p_n$ are positive rational numbers with (2.1) for all $x_1, \ldots x_n \in X$, if and only if $f(x) = A(x) + f(0)$ for all $x \in X$, where $A : X \to Y$ is additive function and $f(0)$ is a constant.*

Proof. (Neccessity) Suppose $f : X \to Y$ satisfies the functional equation (pJ). Define a function $g : X \to Y$ by

$$g(x) = f(x) - f(0),$$

for all $x \in X$. Note that $g(0) = 0$.

Consider

$$g\left(\sum_{i=1}^{n} p_i x_i\right) = f\left(\sum_{i=1}^{n} p_i x_i\right) - f(0) = \sum_{i=1}^{n} p_i f(x_i) - f(0)$$
$$= \sum_{i=1}^{n} p_i g(x_i). \tag{2.2}$$

Note that g satisfies (pJ). Let $s \in \{1, \dots n\}$. By setting $x_s = x$ and $x_1 = \dots = x_{s-1} = x_{s+1} = \dots = x_n = 0$, (2.2) becomes

$$g(p_s x) = p_s g(x), \tag{2.3}$$

for all $s \in \{1, \dots n\}$ and for all $x \in X$. Next, we put $x_s = x, x_{s+1} = y$ and $x_1 = \dots = x_{s-1} = x_{s+2} = \dots = x_n = 0$ in (2.2) and using (2.3), we will have

$$g(p_s x + p_{s+1} y) = g(p_s x) + g(p_{s+1} y),$$

for all $x, y \in X$. Therefore g is additive function. By definition of g, we get $f(x) = g(x) + f(0)$ for all $x \in X$.

(Sufficiency) Suppose that $f(x) = A(x) + f(0)$ for all $x \in X$, where $A : X \to Y$ is an additive function and $f(0)$ is a constant. Therefore,

$$f\left(\sum_{i=1}^{n} p_i x_i\right) = A\left(\sum_{i=1}^{n} p_i x_i\right) + f(0) = A\left(\sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} p_i f(0)$$
$$= \sum_{i=1}^{n} p_i (A(x_i) + f(0)) = \sum_{i=1}^{n} p_i f(x_i).$$

Hence, this completes the proof. $\square$

2.2 Generalized Stability on an n-Dimensional Functional Equation of Jensen Type

In this section, we will study the generalized stability of (pJ) as the following theorem.

Theorem 2.2.1. *Let $\phi : X^n \to [0, \infty)$ be a function. For each integer $s = 1, \ldots, n$, let $\phi_s : X \to [0, \infty)$ be a function such that*

$$\phi_s(x) = \phi(\underbrace{0, \ldots, 0}_{s-1}, x, \underbrace{0, \ldots, 0}_{n-s}). \tag{2.4}$$

Suppose that $\sum\limits_{i=0}^{\infty} p_s^{-i}\phi(p_s^i x)$ converges and $\lim\limits_{m\to\infty} p_s^{-m}\phi(p_s^m x_1, \ldots, p_s^m x_n) = 0$ for all $x_1, \ldots, x_n \in X$. If a function $f : X \to Y$ satisfies the inequality

$$\left\| \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \right\| \leq \phi(x_1, \ldots, x_n), \tag{2.5}$$

for all $x_1, \ldots, x_n \in X$, then there exists a unique function $L : X \to Y$ that satisfies functional equation (pJ) and the inequality

$$\big\| f(x) - L(x) \big\| \leq \sum_{i=0}^{\infty} p_s^{-i-1}\phi_s(p_s^i x), \tag{2.6}$$

for all $x \in X$. The function L is given by

$$L(x) = f(0) + \lim_{m\to\infty} p_s^{-m}\big(f(p_s^m x) - f(0)\big), \tag{2.7}$$

for all $x \in X$.

Proof. Suppose $f : X \to Y$ satisfies the inequality (2.5). Define a function $g : X \to Y$ by

$$g(x) = f(x) - f(0), \tag{2.8}$$

for all $x_1, \ldots, x_n \in X$. It should be noted that $g(0) = 0$. By (2.1) and (2.5), we get

$$\left\| \sum_{i=1}^{n} p_i g(x_i) - g\left(\sum_{i=1}^{n} p_i x_i \right) \right\| \leq \phi(x_1, \ldots, x_n), \tag{2.9}$$

for all $x_1, \ldots, x_n \in X$. Let $s \in \{1, \ldots, n\}$. By setting $x_s = x$ and $x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0$, (2.9) becomes

$$\left\| p_s g(x) - g(p_s x) \right\| \leq \phi_s(x), \tag{2.10}$$

for all $x \in X$. We can rewrite the above equation in the form of

$$\left\| g(x) - p_s^{-1} g(p_s x) \right\| \leq p_s^{-1} \phi_s(x), \tag{2.11}$$

for all $x \in X$. For each positive integer m and each $x \in X$, we have

$$\begin{aligned} \left\| g(x) - p_s^{-m} g(p_s^m x) \right\| &= \left\| \sum_{i=0}^{m-1} \left(p_s^{-i} g(p_s^i x) - p_s^{-(i+1)} g(p_s^{i+1} x) \right) \right\| \\ &\leq \sum_{i=0}^{m-1} \left\| p_s^{-i} g(p_s^i x) - p_s^{-(i+1)} g(p_s^{i+1} x) \right\| \\ &= \sum_{i=0}^{m-1} p_s^{-i} \left\| g(p_s^i x) - p_s^{-1} g(p_s p_s^i x) \right\| \\ &\leq \sum_{i=0}^{m-1} p_s^{-i-1} \phi_s(p_s^i x). \end{aligned} \tag{2.12}$$

Next, consider the sequence $\{p_s^{-m} g(p_s^m x)\}$. For each positive integers $k < l$ and each $x \in X$,

$$\begin{aligned} \left\| p_s^{-k} g(p_s^k x) - p_s^{-l} g(p_s^l x) \right\| &= p_s^{-k} \left\| g(p_s^k x) - p_s^{-(l-k)} g(p_s^{l-k} p_s^k x) \right\| \\ &\leq p_s^{-k} \sum_{i=0}^{l-k-1} p_s^{-i-1} \phi_s(p_s^{i+k} x) \\ &\leq p_s^{-k-1} \sum_{i=0}^{\infty} p_s^{-i} \phi_s(p_s^{i+k} x). \end{aligned}$$

Since $\sum\limits_{i=0}^{\infty} p_s^{-i}\phi(p_s^i x)$ converges, $\lim\limits_{k\to\infty} p_s^{-k-1}\sum\limits_{i=0}^{\infty} p_s^{-i}\phi_s(p_s^{i+k}x) = 0$. Therefore,

$$L(x) = f(0) + \lim_{m\to\infty} p_s^{-m} g(p_s^m x) \tag{2.13}$$

is well-defined in the Banach space Y. Moreover, as $m \to \infty$, (2.12) becomes

$$\big\|g(x) + f(0) - L(x)\big\| \leq \sum_{i=0}^{\infty} p_s^{-i-1}\phi_s(p_s^i x).$$

By the definition of $g(x)$, we see that inequality (2.6) is valid.

To show that L indeed satisfies (pJ), we replace each x_i in (2.9) with $p_s^m x_i$ and get

$$\left\| \sum_{i=1}^{n} p_i g(p_s^m x_i) - g\left(p_s^m \sum_{i=1}^{n} p_i x_i \right) \right\| \leq \phi(p_s^m x_1, \ldots, p_s^m x_n). \tag{2.14}$$

If we multiply the above inequality by p_s^{-m} and take the limit as $m \to \infty$, then by the definition of L in (2.14) and (2.1), we obtain

$$\left\| \sum_{i=1}^{n} p_i L(x_i) - L\left(\sum_{i=1}^{n} p_i x_i \right) \right\| \leq \lim_{m\to\infty} p_s^{-m}\phi(p_s^m x_1, \ldots, p_s^m x_n) = 0, \tag{2.15}$$

which implies that

$$\sum_{i=1}^{n} p_i L(x_i) = L\left(\sum_{i=1}^{n} p_i x_i \right), \tag{2.16}$$

for all $x_1, \ldots, x_n \in X$.

To prove the uniqueness, suppose there is another function $L' : X \to Y$ satisfying (pJ) and (2.6). Observe that if we replace x_s by x and put $x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0$ in (2.16), we get

$$p_s L(x) + (1 - p_s)L(0) = L(p_s x), \tag{2.17}$$

for all $x \in X$, and

$$L(0) = f(0) + \lim_{m\to\infty} p_s^{-m} g(0) = f(0).$$

The function $L^{'}$ obviously possesses the same properties. Therefore,

$$p_s\big(L(x) - L^{'}(x)\big) = L(p_s x) - L^{'}(p_s x) \tag{2.18}$$

for all $x \in X$. We can prove by the mathematical induction that for each positive integer m,

$$p_s^m\big(L(x) - L^{'}(x)\big) = L(p_s^m x) - L^{'}(p_s^m x),$$

for all $x \in X$. Therefore, for each positive integer m,

$$\begin{aligned}\big\|L(x) - L^{'}(x)\big\| &= p_s^{-m}\big\|L(p^m x) - L^{'}(p_s^m x)\big\| \\ &\leq p_s^{-m}\Big(\big\|L(p^m x) - f(p_s^m x)\big\| + \big\|L^{'}(p_s^m x) - f(p_s^m x)\big\|\Big) \\ &\leq 2p_s^{-m}\sum_{i=0}^{\infty} p_s^{-i-1}\phi_s(p_s^{i+m} x),\end{aligned}$$

for all $x \in X$. Since $\sum\limits_{i=0}^{\infty} p_s^{-i}\phi(p_s^i x)$ converges, $\lim\limits_{m\to\infty} p_s^{-m}\sum\limits_{i=0}^{\infty} p_s^{-i-1}\phi(p_s^{i+m}x) = 0$. We conclude that $L(x) = L^{'}(x)$ for all $x \in X$. □

Theorem 2.2.2. *Let $\phi : X^n \to [0,\infty)$ be a function. For each integer $s = 1,\ldots,n$, let $\phi_s : X \to [0,\infty)$ be a function such that (2.4) and $\sum\limits_{i=0}^{\infty} p_s^i\phi(p_s^{-i}x)$ converges and $\lim\limits_{m\to\infty} p_s^m\phi(p_s^{-m}x_1,\ldots,p_s^{-m}x_n) = 0$ for all $x_1,\ldots,x_n \in X$.*
If a function $f : X \to Y$ satisfies the inequality (2.5), then there exists a unique function $L : X \to Y$ that satisfies functional equation (pJ) and the inequality

$$\big\|f(x) - L(x)\big\| \leq \sum_{i=1}^{\infty} p_s^{i-1}\phi_s(p_s^{-i}x) \tag{2.19}$$

for all $x \in X$. The function L is given by

$$L(x) = f(0) + \lim_{m\to\infty} p_s^m f(p_s^{-m}x) \tag{2.20}$$

for all $x \in X$.

Proof. Let $f : X \to Y$ satisfy the inequality (2.5). Similar to the derivation of (2.8)-(2.11), we can replace inequality (2.11) with

$$\left\|g(x) - p_s g(p_s^{-1}x)\right\| \leq \phi_s(p_s^{-1}x),$$

for all $x \in X$. For each positive integer m and each $x \in X$, we get

$$\begin{aligned}
\left\|g(x) - p_s^m g(p_s^{-m}x)\right\| &= \left\|\Big(\sum_{i=1}^{m} p_s^{i-1} g(p_s^{-(i-1)}x) - p_s^i g(p_s^{-i}x)\Big)\right\| \\
&\leq \sum_{i=1}^{m} \left\|p_s^{i-1} g(p_s^{-(i-1)}x) - p_s^i g(p_s^{-i}x)\right\| \\
&= \sum_{i=1}^{m} p_s^{i-1} \left\|g(p_s^{-(i-1)}x) - p_s g(p_s^{-1}p_s^{-(i-1)}x)\right\| \\
&\leq \sum_{i=1}^{m} p_s^{i-1} \phi_s(p_s^{-i}x). \qquad (2.21)
\end{aligned}$$

We now investigate the sequence $\{p_s^m g(p_s^{-m}x)\}$. For each positive integer $k < l$ and each $x \in X$,

$$\begin{aligned}
\left\|p_s^k g(p_s^{-k}x) - p_s^l g(p_s^{-l}x)\right\| &= p_s^k \left\|g(p_s^{-k}x) - p_s^{l-k} g(p_s^{-(l-k)}p_s^{-k}x)\right\| \\
&\leq p_s^k \sum_{i=1}^{l-k} p_s^{i-1} \phi_s(p_s^{-i-k}x) \\
&\leq p_s^{k-1} \sum_{i=1}^{\infty} p_s^i \phi_s(p_s^{-i-k}x).
\end{aligned}$$

Since $\sum_{i=0}^{\infty} p_s^i \phi(p_s^{-i}x)$ converges, $\lim_{k\to\infty} p_s^{k-1} \sum_{i=0}^{\infty} p_s^i \phi_s(p_s^{-i-k}x) = 0$. Thus,

$$L(x) = f(0) + \lim_{m\to\infty} p_s^m g(p_s^{-m}x) \qquad (2.22)$$

is well-defined in the Banach space Y. Furthermore, (2.21) becomes as $m \to \infty$,

$$\left\|g(x) + f(0) - L(x)\right\| \leq \sum_{i=1}^{\infty} p_s^{i-1} \phi_s(p_s^{-i}x).$$

By the definition of $g(x)$, inequality (2.19) is valid.

In order to show that L satisfies (pJ), we replace each x_i by $p_s^{-m}x_i$ in (2.9) and multiply p_s^m. By taking the limit as $m \to \infty$, we have

$$\left\| \sum_{i=1}^{n} p_i L(x_i) - L\left(\sum_{i=1}^{n} p_i x_i \right) \right\| \leq \lim_{m\to\infty} p_s^m \phi(p_s^{-m}x_1, \ldots, p_s^{-m}x_n) = 0,$$

which implies (2.16).

To prove the uniqueness, suppose there is another function $L^{'} : X \longrightarrow Y$ satisfying (pJ) and (2.6). Replacing x_s by $p_s^{-1}x$ and put $x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0$ in (2.16), (2.18) becomes

$$p_s\big(L(p_s^{-1}x) - L^{'}(p_s^{-1}x)\big) = L(x) - L^{'}(x).$$

For each positive m, we can show by mathematical induction that

$$p_s^m\big(L(p_s^{-m}x) - L^{'}(p_s^{-m}x)\big) = L(x) - L^{'}(x),$$

for all $x \in X$. Hence, for each positive integer m,

$$\begin{aligned}\big\|L(x) - L^{'}(x)\big\| &= p_s^m\big\|L(p^{-m}x) - L^{'}(p_s^{-m}x)\big\| \\ &\leq p_s^m\big(\big\|L(p^{-m}x) - f(p_s^{-m}x)\big\| + \big\|L^{'}(p_s^{-m}x) - f(p_s^{-m}x)\big\|\big) \\ &\leq 2p_s^m \sum_{i=1}^{\infty} p_s^{i-1}\phi_s(p_s^{-i-m}x),\end{aligned}$$

for all $x \in X$. Since $\sum\limits_{i=1}^{\infty} p_s^i\phi(p_s^{-i}x)$ converges, $\lim\limits_{m\to\infty} p_s^m \sum\limits_{i=0}^{\infty} p_s^{i-1}\phi(p_s^{-i-m}x) = 0$. We therefore obtain that $L(x) = L^{'}(x)$ for all $x \in X$. □

2.3 Stability on an n-Dimensional Functional Equation of Jensen Type

In this section, we will give the stability of (pJ) in various case as in the following theorem.

Theorem 2.3.1. *Let $\varepsilon > 0$ be a real number. If a function $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i \right) \right\| \leq \varepsilon, \tag{2.23}$$

for all $x_1, \ldots, x_n \in X$, then there exists a unique function $L : X \to Y$ that satisfies (pJ) and

$$\left\| f(x) - L(x) \right\| \leq \frac{\varepsilon}{1 - p_{\min}},$$

for all $x \in X$, where $p_{\min} = \min\{p_1, \ldots, p_n\}$.

Proof. Let

$$\phi(x_1, \ldots, x_n) = \varepsilon$$

for all $x_1, \ldots, x_n \in X$ in Theorem 2.2.2. We note that Theorem 2.2.2 holds for every $s = 1, \ldots, n$. We then choose s such that $p_s = p_{\min} = \min\{p_1, \ldots, p_n\}$. Therefore, (2.19) becomes

$$\left\| f(x) - L(x) \right\| \leq \varepsilon \sum_{i=1}^{\infty} p_s^{i-1} = \frac{\varepsilon}{1 - p_s} = \frac{\varepsilon}{1 - p_{\min}},$$

for all $x \in X$ as desired. $\square$

The following theorem proves the stability of (pJ).

Theorem 2.3.2. *Let $\varepsilon > 0$ and $r > 0$ be real numbers with $r \neq 1$. If a function $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i \right) \right\| \leq \varepsilon \sum_{i=1}^{n} \|x_i\|^r, \tag{2.24}$$

for all $x_1, \ldots, x_n \in X$, then there exists a unique function $L : X \to Y$ that satisfies (pJ) and

$$\left\| f(x) - L(x) \right\| \leq \frac{\varepsilon}{M} \|x\|^r,$$

for all $x \in X$, where $M = \max_{i=1,\ldots,n} \mid p_i - p_i^r \mid$.

Proof. For the case $0 < r < 1$, we let

$$\phi(x_1, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^r,$$

for all $x_1, \ldots, x_n \in X$ in Theorem 2.2.2. We note that Theorem 2.2.2 holds for every $s = 1, \ldots, n$. We then choose s such that

$$\mid p_s - p_s^r \mid = M = \max_{i=1,\ldots,n} \mid p_i - p_i^r \mid .$$

Thus, (2.6) becomes

$$\begin{aligned}\big\|f(x) - L(x)\big\| &\leq \varepsilon \sum_{i=1}^{\infty} p_s^{i-1} \|p_s^{-i} x\|^r = \varepsilon \|x\|^r p_s^{-1} \sum_{i=1}^{\infty} p_s^{i(1-r)} \\ &= \varepsilon p_s^{-1} \|x\|^r \Big(\frac{p_s^{1-r}}{1 - p_s^{1-r}} \Big) = \frac{\varepsilon}{p_s^r - p_s} \|x\|^r = \frac{\varepsilon}{M} \|x\|^r,\end{aligned}$$

for all $x \in X$. For the case $r > 1$, we let

$$\phi(x_1, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^r,$$

for all $x_1, \ldots, x_n \in X$ in Theorem 2.1.1. Since Theorem 2.1.1 holds for every $s = 1, \ldots, n$, (2.6) becomes

$$\begin{aligned}\big\|f(x) - L(x)\big\| &\leq \varepsilon \sum_{i=0}^{\infty} p_s^{-i-1} \|p_s^{i} x\|^r = \varepsilon \|x\|^r p_s^{-1} \sum_{i=0}^{\infty} p_s^{i(r-1)} \\ &= \varepsilon \|x\|^r p_s^{-1} \Big(\frac{1}{1 - p_s^{r-1}} \Big) = \frac{\varepsilon}{p_s - p_s^r} \|x\|^r = \frac{\varepsilon}{M} \|x\|^r,\end{aligned}$$

for all $x \in X$. This completes the proof. $\square$

CHAPTER III

THE WEAK FORM OF ALTERNATIVE JENSEN FUNCTIONAL EQUATIONS

In this chapter, we will give a criterion for the existence of the general solution for the functional equation

$$f(xy^{-1}) - 2f(x) + f(xy) = 0 \quad \text{or} \quad f(xy^{-1}) + \beta f(x) + f(xy) = 0. \tag{WA}$$

For the case β is an integer and $\beta \neq -2$, our work appears in [18]. Let $(G, \cdot)$ be a group, $(H, +)$ be a uniquely divisible abelian group. Given a rational number $\beta \neq -2$ and a function $f : G \to H$. For every pair of $x, y \in G$, we define

$$F_y^{(\beta)}(x) := f(xy^{-1}) + \beta f(x) + f(xy),$$

and

$$J_y(x) := f(xy^{-1}) - 2f(x) + f(xy).$$

In addition, we denote the statement

$$\mathcal{P}f_y^{(\beta)}(x) := \Big(J_y(x) = 0 \quad \text{or} \quad F_y^{(\beta)}(x) = 0 \Big).$$

The set of solution to the statement $\mathcal{P}f_y^{(\beta)}(x)$ will be denoted by $\mathcal{A}_{(G,H)}^{(\beta)}$, i.e.,

$$\mathcal{A}_{(G,H)}^{(\beta)} := \{ f : G \to H \mid \mathcal{P}f_y^{(\beta)}(x) \ \text{ for all } x, y \in G \},$$

while the set of solution of $J_y(x) = 0$ is denoted by

$$\mathcal{J}_{(G,H)} := \{f : G \to H \mid J_y(x) = 0 \text{ for all } x, y \in G\}.$$

3.1 Auxiliary Lemmas

Lemma 3.1.1. *Let $f \in \mathcal{A}^{(\beta)}_{(G,H)}$ and $x, y \in G$.*
$J_y(x) = 0$ and $F^{(\beta)}_y(x) = 0$ if and only if $f(x) = 0$.

Proof. Assume that $J_y(x) = 0$ and $F^{(\beta)}_y(x) = 0$. Therefore, $F^{(\beta)}_y(x) - J_y(x) = 0$, i.e.,

$$(\beta + 2)f(x) = 0.$$

Since $\beta \neq -2$, we must have $f(x) = 0$. Conversely, we assume that $f(x) = 0$. Since $f \in \mathcal{A}^{(\beta)}_{(G,H)}$, we have $J_y(x) = 0$ or $F^{(\beta)}_y(x) = 0$. As $f(x) = 0$, therefore $J_y(x) = 0$ and $F^{(\beta)}_y(x) = 0$. □

Lemma 3.1.2. *Let $f \in \mathcal{A}^{(\beta)}_{(G,H)}$ and let $x, y \in G$.*

(1) If $J_y(xy^{-1}) = 0$ and $J_y(xy) = 0$, then $J_y(x) = 0$.

(2) If $F^{(\beta)}_y(xy^{-1}) = 0$ and $F^{(\beta)}_y(xy) = 0$, then $F^{(\beta)}_y(x) = 0$.

Proof. Suppose that all the assumptions in the lemma hold.

(1) If $J_y(x) \neq 0$, then $F^{(\beta)}_y(x) = 0$. Therefore, $J_y(xy^{-1}) + 2F^{(\beta)}_y(x) + J_y(xy) = 0$, i.e,

$$f(xy^{-2}) + 2(1 + \beta)f(x) + f(xy^2) = 0. \tag{3.1}$$

Consider $\mathcal{P}f^{(\beta)}_{y^2}(x)$. The alternative $f(xy^{-2}) - 2f(x) + f(xy^2) = 0$ and (3.1) gives $(2 + \beta)f(x) = 0$, while the alternative $f(xy^{-2}) + \beta f(x) + f(xy^2) = 0$ and (3.1) also gives $(2 + \beta)f(x) = 0$. Since $\beta \neq -2$, we get $f(x) = 0$. Therefore, $J_y(x) = 0$ by Lemma 3.1.1.

(2) If $F_y^{(\beta)}(x) \neq 0$, then $J_y(x) = 0$. Therefore, $F_y^{(\beta)}(xy^{-1}) - \beta J_y(x) + F_y^{(\beta)}(xy) = 0$, i.e.,

$$f(xy^{-2}) + 2(1+\beta)f(x) + f(xy^2) = 0.$$

By a similar argument as above, we will have $f(x) = 0$. Therefore, $F_y^{(\beta)}(x) = 0$ by Lemma 3.1.1.

□

Lemma 3.1.3. *Let $f \in \mathcal{A}_{(G,H)}^{(\beta)}$ and let $x, y \in G$.*
If $J_y(xy^{-1}) = 0$, $J_y(x) \neq 0$ and $F_y^{(\beta)}(xy^2) \neq 0$, then $\beta = 0$ and

$$f(xy^n) = \begin{cases} -f(x) & \text{if } n \in \mathbb{Z}^+, \\ f(x) & \text{if } n \in \mathbb{Z}^-. \end{cases}$$

Proof. Suppose that all the assumptions in the lemma hold. By Lemma 3.1.1, $J_y(x) \neq 0$ implies that $f(x) \neq 0$. From $J_y(x) \neq 0$ and $\mathcal{P}f_y^{(\beta)}(x)$, we obtain that $F_y^{(\beta)}(x) = 0$. From $F_y^{(\beta)}(xy^2) \neq 0$ and $\mathcal{P}f_y^{(\beta)}(xy^2)$, we get $J_y(xy^2) = 0$. From $J_y(xy^{-1}) = 0$ and $J_y(x) \neq 0$, Lemma 3.1.2 gives $J_y(xy) \neq 0$. By the alternatives in $\mathcal{P}f_y^{(\beta)}(xy)$, we obtain that $F_y^{(\beta)}(xy) = 0$. If $F_y^{(\beta)}(xy^3) = 0$, then by Lemma 3.1.2, we must have $F_y^{(\beta)}(xy^2) = 0$, a contradiction. Therefore, $F_y^{(\beta)}(xy^3) \neq 0$ and the alternatives in $\mathcal{P}f_y^{(\beta)}(xy^3)$ give $J_y(xy^3) = 0$.

Eliminating $f(xy^{-1})$ from $J_y(xy^{-1}) = 0$ and $F_y^{(\beta)}(x) = 0$, we get

$$f(xy^{-2}) + (1+2\beta)f(x) + 2f(xy) = 0. \tag{3.2}$$

We will consider each alternative in $\mathcal{P}f_{y^2}^{(\beta)}(x)$ as follows:

(1) Assume that $J_{y^2}(x) = 0$. Solving $F_y^{(\beta)}(xy) = 0$, $J_{y^2}(x) = 0$ and (3.2), we have

$$f(xy) + 2f(x) = 0 \text{ and } f(xy^2) + (1-2\beta)f(x) = 0. \tag{3.3}$$

By (3.3) and $F_y^{(\beta)}(x) = 0$, we get

$$f(xy^{-1}) + (\beta - 2)f(x) = 0. \tag{3.4}$$

From $J_y(xy^2) = 0$ and (3.3), we obtain that $f(xy^3) - 4\beta f(x) = 0$. Considering the alternatives in $\mathcal{P}f_{y^2}^{(\beta)}(xy)$, we conclude that $(\beta + 2)f(x) = 0$. Since $\beta \neq -2$, we must have $f(x) = 0$, a contradiction. Thus this case does not exist.

(2) Assume that $F_{y^2}^{(\beta)}(x) = 0$. Solving $F_y^{(\beta)}(xy) = 0$, $F_{y^2}^{(\beta)}(x) = 0$, and (3.2), we have

$$f(x) + f(xy) = 0 \text{ and } f(xy^2) + (1 - \beta)f(x) = 0. \tag{3.5}$$

Eliminating $f(xy^2)$ and $f(xy^3)$ from $J_y(xy^2) = 0$, $J_y(xy^3) = 0$ and (3.5), we get

$$f(xy^4) + (1 - 3\beta)f(x) = 0. \tag{3.6}$$

We will consider the alternative in $\mathcal{P}f_{y^2}^{(\beta)}(xy^2)$ as follows: If $J_{y^2}(xy^2) = 0$, then from (3.5), (3.6) and $J_{y^2}(xy^2) = 0$, we conclude that $(\beta + 2)f(x) = 0$. Since $\beta \neq -2$, we have $f(x) = 0$, a contradiction. Thus we must get $F_{y^2}^{(\beta)}(xy^2) = 0$. Solving (3.5), (3.6) and $F_{y^2}^{(\beta)}(xy^2) = 0$, we conclude that $(\beta + 2)\beta f(x) = 0$. Since $\beta \neq -2$ and $f(x) \neq 0$, we must have $\beta = 0$ and so $F_{y^2}^{(0)}(x) = 0$.

We already have $\beta = 0$. From (3.5), (3.6) and $J_y(xy^2) = 0$, we obtain that

$$f(xy^n) = -f(x) \text{ for all } n = 1, \ldots, 4. \tag{3.7}$$

We will prove that (3.7) also holds for all $n \in \mathbb{Z}^+$. It is only left to prove that $J_y(xy^n) = 0$ for all $n \geq 4$. We will show this by contradiction. Suppose that $J_y(xy^m) \neq 0$ for some $m \geq 4$. We further assume that m is the least number. Thus the alternatives in $\mathcal{P}f_y^{(0)}(xy^i)$ for each $n = 3, \ldots, m - 1$

give $J_y(xy^i) = 0$, i.e.,

$$f(xy^{i-1}) - 2f(xy^i) + f(xy^{i+1}) = 0 \text{ for all } n = 3, \ldots, m-1. \tag{3.8}$$

From (3.8) and $f(xy^3) = f(xy^4) = -f(x)$ in (3.7), we have

$$f(xy^n) = -f(x) \text{ for all } n = 1, \ldots, m. \tag{3.9}$$

The alternatives in $\mathcal{P}f_y^{(0)}(xy^m)$ and $J_y(xy^m) \neq 0$ give $F_y^{(0)}(xy^m) = 0$. By $F_y^{(0)}(xy^m) = 0$ and $f(xy^{m-1}) = -f(x)$ in (3.9), we get $f(xy^{m+1}) = f(x)$.

(a) If $m = 2k$ for some $k \geq 2$, then $f(xy^{2k+1}) = f(xy^{m+1}) = f(x)$. From $F_y^{(0)}(x) = 0$ and $f(xy) = -f(x)$ in (3.9), we obtain that $f(xy^{-1}) = f(x)$. Since $2 \leq k < m$, $f(xy^k) = -f(x)$ by (3.9). The alternatives in $\mathcal{P}f_{y^{k+1}}^{(0)}(xy^k)$ give $f(x) = 0$, a contradiction.

(b) If $m = 2k+1$ for some $k \geq 2$, then $f(xy^{2k+2}) = f(xy^{m+1}) = f(x)$. Since $2 \leq k < m$, $f(xy^{k+1}) = -f(x)$ by (3.9). The alternatives in $\mathcal{P}f_{y^{k+1}}^{(0)}(xy^{k+1})$ give $f(x) = 0$, a contradiction.

Thus $f(xy^n) = -f(x)$ for all $n \in \mathbb{Z}^+$ as desired.

Since $F_y^{(0)}(x) = 0, F_{y^2}^{(0)}(x) = 0$ and $f(xy) = f(xy^2) = -f(x)$ in (3.7), we have $f(xy^{-2}) = f(xy^{-1}) = f(x)$. We can repeat the above process to show that $f(xy^n) = f(x)$ for all $n \in \mathbb{Z}^-$ by substituting x by xy^{-2} and y by y^{-1} in the previous arguments. $\square$

Lemma 3.1.4. *Let $f \in \mathcal{A}_{(G,H)}^{(\beta)}$ and let $x, y \in G$.*

(1) If $F_y^{(\beta)}(xy^{-1}) = 0, J_y(x) \neq 0$ and $F_y^{(\beta)}(xy) = 0$, then $\beta \in \{1, 2\}$.

(2) If $F_y^{(\beta)}(xy^{-1}) \neq 0$ and $J_y(x) \neq 0$, then $\beta \neq 2$.

Proof. Suppose that all the assumptions in the lemma hold.

(1) From $J_y(x) \neq 0$ and $\mathcal{P}f_y^{(\beta)}(x)$, we have $F_y^{(\beta)}(x) = 0$. Therefore, $f(x) \neq 0$ by Lemma 3.1.1. Observing that $F_y^{(\beta)}(xy^{-1}) - \beta F_y^{(\beta)}(x) + F_y^{(\beta)}(xy) = 0$, which reduces to

$$f(xy^{-2}) + (2 - \beta^2)f(x) + f(xy^2) = 0. \tag{3.10}$$

We will now consider each alternative in $\mathcal{P}f_{y^2}^{(\beta)}(x)$. The alternative $f(xy^{-2}) - 2f(x) + f(xy^2) = 0$ and (3.10) gives $(4-\beta^2)f(x) = 0$, while the alternative $f(xy^{-2})+\beta f(x)+ f(xy^2) = 0$ and (3.10) gives $(2 - \beta - \beta^2)f(x) = 0$. Since $f(x) \neq 0$ and $\beta \neq -2$, we must have $\beta = 1$ or $\beta = 2$.

(2) We will prove $\beta \neq 2$ by contradiction. Suppose that $\beta = 2$. From $F_y^{(2)}(xy^{-1}) \neq 0$ and $\mathcal{P}f_y^{(2)}(xy^{-1})$, we obtain $J_y(xy^{-1}) = 0$. From $J_y(x) \neq 0$ and $\mathcal{P}f_y^{(2)}(x)$, we get $J_y(x) = 0$ and thus $f(x) \neq 0$ by Lemma 3.1.1. Since $F_y^{(2)}(xy^{-1}) \neq 0$ and $F_y^{(2)}(x) = 0$, Lemma 3.1.2 gives $J_y(xy^{-2}) \neq 0$. Thus by the alternatives in $\mathcal{P}f_y^{(2)}(xy^{-2})$, we have $J_y(xy^{-2}) = 0$. From $J_y(xy^{-1}) = 0$ and $J_y(x) \neq 0$, Lemma 3.1.2 gives $J_y(xy) \neq 0$. By the alternatives in $\mathcal{P}f_y^{(2)}(xy)$, we obtain $J_y(xy) = 0$. Eliminating $f(xy^{-1})$ from $J_y(xy^{-1}) = 0$ and $F_y^{(2)}(x) = 0$, we get

$$f(xy^{-2}) + 5f(x) + 2f(xy) = 0. \tag{3.11}$$

Eliminating $f(xy^{-2})$ and $f(xy^2)$ from $F_y^{(2)}(xy) = 0$, (3.11) and each alternative in $\mathcal{P}f_{y^2}^{(2)}(x)$, we obtain that

$$2f(x) + f(xy) = 0 \text{ or } f(x) + f(xy) = 0. \tag{3.12}$$

From $J_y(xy^{-2}) = 0$, $J_y(xy^{-1}) = 0$ and $F_y^{(2)}(x) = 0$, we have $J_y(xy^{-2})+2J_y(xy^{-1})+ 3J_y(x) = 0$, i.e.,

$$f(xy^{-3}) + 8f(x) + 3f(xy) = 0. \tag{3.13}$$

Eliminating $f(xy^{-3})$ from (3.13) and each alternatives in $\mathcal{P}f_{y^2}^{(2)}(xy^{-1})$, we get

$$4f(x) + f(xy) + f(xy^{-1}) = 0 \text{ or } 4f(x) + f(xy) - f(xy^{-1}) = 0. \tag{3.14}$$

Solving $F_y^{(2)}(x) = 0$ and (3.14), we conclude that

$$f(x) = 0 \text{ or } 3f(x) + f(xy) = 0. \tag{3.15}$$

Combining (3.12) and (3.15), we have $f(x) = 0$, a contradiction. Therefore, we must get $\beta = 2$ as desired.

Hence we have the desired result. □

Lemma 3.1.5. *Let $f \in \mathcal{A}_{(G,H)}^{(\beta)}$ and let $x, y \in G$.*
If $F_y^{(\beta)}(xy^{-1}) \neq 0$, $J_y(x) \neq 0$ and $F_y^{(\beta)}(xy^2) = 0$, then $\beta = 1$ and

(1) $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{a}, -2a, \overline{a})$ *for some $a \in H$, or*

(2) $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{-2a, a, \ldots, a})$*, a periodic sequence of an odd period $p \geq 5$, for some $a \in H$.*

Proof. Suppose that all the assumptions in the lemma hold. Thus $\beta \neq 2$ by Lemma 3.1.4. From $F_y^{(\beta)}(xy^{-1}) \neq 0$ and $\mathcal{P}f_y^{(\beta)}(xy^{-1})$, we get $J_y(xy^{-1}) = 0$. From $J_y(x) \neq 0$ and $\mathcal{P}f_y^{(\beta)}(x)$, we have $F_y^{(\beta)}(x) = 0$. Therefore, $f(xy^{-1}) \neq 0$ and $f(x) \neq 0$ by Lemma 3.1.1. Since $J_y(x) \neq 0$ and $J_y(xy^{-1}) = 0$, Lemma 3.1.2 gives $J_y(xy) \neq 0$. By the alternatives in $\mathcal{P}f_y^{(\beta)}(xy)$, we get $F_y^{(\beta)}(xy) = 0$. From $F_y^{(\beta)}(x) = 0, J_y(xy) \neq 0, F_y^{(\beta)}(xy^2) = 0$ and $\beta \neq 2$, Lemma 3.1.4 gives $\beta = 1$. Eliminating $f(xy^{-1})$ from $J_y(xy^{-1}) = 0$ and $F_y^{(1)}(x) = 0$, we obtain

$$f(xy^{-2}) + 3f(x) + 2f(xy) = 0. \tag{3.16}$$

Consider the alternatives in $\mathcal{P}f_{y^2}^{(1)}(x)$ as follows: Solving $J_{y^2}(x) = 0, F_y^{(1)}(xy) = 0$ and (3.16) gives $2f(x) + f(xy) = 0$, while solving $F_{y^2}^{(1)}(x) = 0, F_y^{(1)}(xy) = 0$ and (3.16) gives $f(x) + f(xy) = 0$. If $f(x) + f(xy) = 0$, then $F_y^{(1)}(x) = 0$ simplifies to $f(xy^{-1}) = 0$, a contradiction. Thus we must have $2f(x) + f(xy) = 0$. Let $f(x) = a$. From $J_y(xy^{-1}) = 0, F_y^{(1)}(x) = 0, F_y^{(1)}(xy) = 0, F_y^{(1)}(xy^2) = 0$ and $2f(x) + f(xy) = 0$, we conclude that

$$(f(xy^{-2}), f(xy^{-1}), f(x), f(xy), f(xy^2), f(xy^3)) = (a, a, a, -2a, a, a). \tag{3.17}$$

If $F_y^{(1)}(xy^3) = 0$, then by $f(xy^2) = f(xy^3) = a$ in (3.17), we get $f(xy^4) = -2a$. From $f(xy^{-2}) = a$ and $f(xy) = -2a$ in (3.17), the alternatives in $\mathcal{P}f_{y^3}^{(1)}(xy)$ give $a = 0$, a contradiction. Therefore, $F_y^{(1)}(xy^3) \neq 0$. By the alternatives in $\mathcal{P}f_y^{(1)}(xy^3)$, we have $J_y(xy^3) = 0$, i.e., $f(xy^4) = a$. From the alternatives in $\mathcal{P}f_y^{(1)}(xy^n)$ for each $n \geq 4$, we will consider two possible cases as follows:

(1) Assume that $J_y(xy^n) = 0$ for all $n \geq 4$, i.e.,

$$f(xy^{n-1}) - 2f(xy^n) + f(xy^{n+1}) = 0 \text{ for all } n \geq 4. \tag{3.18}$$

From (3.18), $f(xy^4) = a$ and $f(xy^2) = f(xy^3) = a$ in (3.17), we conclude that

$$f(xy^n) = a \text{ for all } n \geq 2. \tag{3.19}$$

Next, we will show that (3.19) also holds for all $n \leq 0$. It is only left to prove that $f(xy^n) = a$ for all $n \leq -3$. Let $m \leq -3$ be an integer. We will consider the alternatives in $\mathcal{P}f_{y^{m-1}}^{(1)}(xy)$ as follows: From $f(xy) = -2a$ in (3.17) and $f(xy^{-m+2}) = a$ in (3.19), the alternative $F_{y^{m-1}}^{(1)}(xy) = 0$ and (3.19) gives $f(xy^m) = -5a$, while the alternative $F_{y^{m-1}}^{(1)}(xy) = 0$ and (3.19) gives $f(xy^m) = a$. First, assume that $f(xy^m) = -5a$. From $f(x) = a$ in (3.17) and $f(xy^{-m}) = a$ in (3.19), the alternatives in $\mathcal{P}f_{y^m}^{(1)}(x)$ give $a = 0$, a contradiction. Thus we must have $f(xy^m) = a$. Therefore, we get that (3.19) holds for all $n \leq 0$ and so $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{a}, -2a, \overline{a})$.

(2) Assume that there exists $m \geq 4$ such that $J_y(xy^m) \neq 0$. Thus we can further assume that m is the least number. From the alternatives in $\mathcal{P}f_y^{(1)}(xy^n)$ for each $3 \leq n \leq m-1$, we have $J_y(xy^n) = 0$, i.e.,

$$f(xy^{n-1}) - 2f(xy^n) + f(xy^{n+1}) = 0 \text{ for all } 3 \leq n \leq m-1. \tag{3.20}$$

Since (3.20) and $f(xy^2) = f(xy^3) = a$ in (3.17), we obtain that

$$f(xy^n) = a \text{ for all } 2 \leq n \leq m. \tag{3.21}$$

From $J_y(xy^m) \neq 0$ and $\mathcal{P}f_y^{(1)}(xy^m)$, we get $F_y^{(1)}(xy^m) = 0$. Since $f(xy^{m-1}) = f(xy^m) = a$ in (3.21), $F_y^{(1)}(xy^m) = 0$ reduces to $f(xy^{m+1}) = -2a$. Next, we will show that m must be odd by contradiction. Suppose that $m = 2k$ for some $k \in \mathbb{Z}$. We have $f(xy^{2k+1}) = f(xy^{m+1}) = -2a$. From $f(xy) = -2a$ in (3.17) and $f(xy^{k+1}) = a$ in (3.21), the alternatives in $\mathcal{P}f_{y^k}^{(1)}(xy^{k+1})$ give $a = 0$, a contradiction. Thus m must be odd. Next, we will show that

$$(f(xy^{m+2}), f(xy^{m+3}), \ldots, f(xy^{2m}), f(xy^{2m+1})) = (a, a, \ldots, a, -2a). \tag{3.22}$$

Let p be an integer with $1 \leq p \leq m-1$. From $f(xy^{m-p+1}) = a$ in (3.21) and $f(xy^{m+1}) = -2a$, the alternatives in $\mathcal{P}f_{y^p}^{(1)}(xy^{m+1})$ give $f(xy^{m+p+1}) = -5a$ or $f(xy^{m+p+1}) = a$. First, assume that $f(xy^{m+p+1}) = -5a$. Since $0 \leq m-p-1 \leq m-2$, by (3.17) and (3.21), we have $f(xy^{m-p-1}) = -2a$ or $f(xy^{m-p-1}) = a$. From $f(xy^m) = a$ in (3.21), the alternatives in $\mathcal{P}f_{y^{p+1}}^{(1)}(xy^m)$ give $a = 0$, a contradiction. Thus we must have $f(xy^{m+p+1}) = a$. Therefore, $f(xy^n) = a$ for all $m+2 \leq n \leq 2m$. Considering the alternatives in $\mathcal{P}f_y^{(1)}(xy^{2m})$, we get $f(xy^{2m+1}) = a$ or $f(xy^{2m+1}) = -2a$. First, assume that $f(xy^{2m+1}) = a$. From $f(xy) = f(xy^{m+1}) = -2a$, the alternatives in $\mathcal{P}f_{y^m}^{(1)}(xy^{m+1})$ give $a = 0$ and therefore a contradiction. Thus we must have $f(xy^{2m+1}) = -2a$.

Similarly, by repeating the process of (3.22), we obtain that

$$(f(xy^{2m+2}), f(xy^{2m+3}), \ldots, f(xy^{3m}), f(xy^{3m+1})) = (a, a, \ldots, a, -2a)$$

and so on. Eventually, we arrive that

$$\left(f(xy^{im+2}), f(xy^{im+3}), \ldots, f(xy^{(i+1)m}), f(xy^{(i+1)m+1})\right) = (a, a, \ldots, a, -2a)$$

for all $i \geq 0$. Moreover, we can similarly repeat the process of (3.22) for each $f(xy^k)$ with $k \leq -3$ to get $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{-2a, a, \ldots, a})$, a periodic sequence of an odd period $p \geq 5$ as desired.

The proof is now complete. $\square$

3.2 Main Results and Some Examples

In this section, all lemmas in the previous section will be consolidated to provide three more lemmas, which will eventually comprise our main theorem.

First, we will make the following crucial observations. Let $f \in \mathcal{A}^{(\beta)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$ and let $x, y \in G$. By the definition of $\mathcal{A}^{(\beta)}_{(G,H)}$ and $\mathcal{J}_{(G,H)}$, one of the following properties holds:

(1) $J_y(xy^n) \neq 0$ for all $n \in \mathbb{Z}$.

(2) There exists $m \in \mathbb{Z}$ such that

(2.1) $J_y(xy^m) \neq 0$ and $J_y(xy^{m-1}) = 0$, or

(2.2) $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$.

The above observation will be used in the proof of the lemmas and theorem as follows.

Lemma 3.2.1. *Let $f \in \mathcal{A}^{(0)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$ and let $x, y \in G$.*
Then $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{-a}, \overline{a})$ for some $a \in H$.

Proof. Suppose that all the assumptions in the lemma hold. By the above observation, we have the following cases:

(1) Assume that $J_y(xy^n) \neq 0$ for all $n \in \mathbb{Z}$. The alternatives in $\mathcal{P}f^{(0)}_y(xy^{-1})$ and $\mathcal{P}f^{(0)}_y(xy)$ give $F^{(0)}_y(xy^{-1}) = 0$ and $F^{(0)}_y(xy) = 0$, respectively. From $F^{(0)}_y(xy^{-1}) = 0$, $J_y(x) \neq 0$

and $F_y^{(0)}(xy) = 0$, Lemma 3.1.4 gives a contradiction. Thus the solution does not exist in this case.

(2) Assume that there exists $m \in \mathbb{Z}$ such that $J_y(xy^m) \neq 0$ and $J_y(xy^{m-1}) = 0$, or $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$. Therefore, each case implies that:

(2.1) If $J_y(xy^m) \neq 0$ and $J_y(xy^{m-1}) = 0$, then the alternatives in $\mathcal{P}f_y^{(0)}(xy^m)$ give $F_y^{(0)}(xy^m) = 0$. From $J_y(xy^{m-1}) = 0$ and $J_y(xy^m) \neq 0$, Lemma 3.1.2 gives $J_y(xy^{m+1}) \neq 0$. We will consider the alternatives in $\mathcal{P}f_y^{(0)}(xy^{m+2})$ as follows: If $F_y^{(0)}(xy^{m+2}) = 0$, then from $J_y(xy^{m+1}) \neq 0$ and $F_y^{(0)}(xy^m) = 0$, we get a contradiction by Lemma 3.1.4. Thus we must have $F_y^{(0)}(xy^{m+2}) \neq 0$. From $J_y(xy^{m-1}) = 0$, $J_y(xy^m) \neq 0$ and $F_y^{(0)}(xy^{m+2}) \neq 0$, we get

$$f(xy^{m+n}) = \begin{cases} -f(xy^m) & \text{if } n \in \mathbb{Z}^+, \\ f(xy^m) & \text{if } n \in \mathbb{Z}^- \end{cases}$$

by substituting x by xy^m in Lemma 3.1.3. Therefore,

$$\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{f(xy^m)}, \overline{-f(xy^m)}).$$

(2.2) If $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$, then we conclude that $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{-f(xy^m)}, \overline{f(xy^m)})$ by substituting x by xy^{2m} and y by y^{-1} in the arguments in the case 2.1.

All the above consideration completes the proof. $\square$

Lemma 3.2.2. *Let $f \in \mathcal{A}^{(1)}_{(G,H)} \backslash \mathcal{J}_{(G,H)}$ and let $x, y \in G$. Then*

(1) $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{a, b, -a-b})$ *for some $a, b \in H$, or*

(2) $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{a}, -2a, \overline{a})$ *for some $a \in H$, or*

(3) $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{-2a, a, \ldots, a})$*, a periodic sequence of an odd period $p \geq 5$, for some $a \in H$.*

Proof. Suppose that all the assumptions in the lemma hold. By the above observation, we have the following cases:

(1) Assume that $J_y(xy^n) \neq 0$ for all $n \in \mathbb{Z}$. Thus the alternatives in $\mathcal{P}f_y^{(1)}(xy^n)$ give $F_y^{(1)}(xy^n) = 0$, i.e.,

$$f(xy^{n-1}) + f(xy^n) + f(xy^{n+1}) = 0$$

for all $n \in \mathbb{Z}$. This implies the property (1).

(2) Assume that there exists $m \in \mathbb{Z}$ such that $J_y(xy^m) \neq 0$ and $J_y(xy^{m-1}) = 0$, or $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$. Therefore, each case implies that:

(2.1) Suppose $J_y(xy^m) \neq 0$ and $J_y(xy^{m-1}) = 0$. In the case when $F_y^{(1)}(xy^{m-1}) \neq 0$, we will consider the alternatives in $\mathcal{P}f_y^{(1)}(xy^{m+2})$ as follows: If $F_y^{(1)}(xy^{m+2}) \neq 0$, then Lemma 3.1.3 gives a contradiction. Therefore, $F_y^{(1)}(xy^{m+2}) = 0$. From $F_y^{(1)}(xy^{m-1}) \neq 0$, $J_y(xy^m) \neq 0$ and $F_y^{(1)}(xy^{m+2}) = 0$, by substituting x by xy^m in Lemma 3.1.5, we get the property (2) and (3). Thus we will only consider the case when the alternatives in $\mathcal{P}f_y^{(1)}(xy^{-1})$ are equivalent, i.e., we also get $F_y^{(1)}(xy^{m-1}) = 0$. Thus $f(xy^{m-1}) = 0$ by Lemma 3.1.1. From $J_y(xy^{m-1}) = 0$ and $J_y(xy^m) \neq 0$, we get $J_y(xy^{m+1}) \neq 0$ by Lemma 3.1.2. Thus the alternatives in $\mathcal{P}f_y^{(1)}(xy^{m+1})$ give $F_y^{(1)}(xy^{m+1}) = 0$. From $J_y(xy^m) \neq 0$ and $\mathcal{P}f_y^{(1)}(xy^m)$, we have $F_y^{(1)}(xy^m) = 0$. Therefore, $f(xy^m) \neq 0$ by Lemma 3.1.1. Let $f(xy^m) = a$. From $f(xy^{-1}) = 0$, $F_y^{(1)}(xy^m) = 0$ and $F_y^{(1)}(xy^{m+1}) = 0$, we conclude that

$$(f(xy^{m-1}), f(xy^m), f(xy^{m+1}), f(xy^{m+2})) = (0, a, -a, 0). \tag{3.23}$$

By (3.23), the alternatives in $\mathcal{P}f_y^{(1)}(xy^{m+2})$, $\mathcal{P}f_{y^2}^{(1)}(xy^{m+2})$ and $\mathcal{P}f_{y^3}^{(1)}(xy^{m+2})$ give

$$(f(xy^{m+3}), f(xy^{m+4}), f(xy^{m+5})) = (a, -a, 0).$$

Similarly, the alternatives in $\mathcal{P}f_y^{(1)}(xy^{m+5})$, $\mathcal{P}f_{y^2}^{(1)}(xy^{m+5})$ and $\mathcal{P}f_{y^3}^{(1)}(xy^{m+5})$ give

$(f(xy^{m+6}), f(xy^{m+7}), f(xy^{m+8})) = (a, -a, 0)$ and so on. Finally, we get

$$(f(xy^{m+3k}), f(xy^{m+3k+1}), f(xy^{m+3k+2})) = (a, -a, 0). \tag{3.24}$$

for all $k \geq 0$. On the other hand, the alternatives in $\mathcal{P}f_y^{(1)}(xy^{m-1})$, $\mathcal{P}f_{y^2}^{(1)}(xy^{m-1})$ and $\mathcal{P}f_{y^3}^{(1)}(xy^{m-1})$ give $(f(xy^{m-2}), f(xy^{m-3}), f(xy^{m-4})) = (-a, a, 0)$. Repeating the process similar to (3.24), we have

$$(f(xy^{m-3k}), f(xy^{m-3k-1}), f(xy^{m-3k-2})) = (a, 0, -a)$$

for all $k \geq 0$. Therefore, we get the property (1) when $b = -a$.

(2.2) If $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$, then we have the similar results as the case 2.1 by substituting x by xy^{2m} and y by y^{-1}.

Hence we have the desired result. $\square$

Lemma 3.2.3. *Let $f \in \mathcal{A}_{(G,H)}^{(2)} \backslash \mathcal{J}_{(G,H)}$ and let $x, y \in G$.*
Then $f(xy^n) = (-1)^n\big(f(x) - n(f(x) + f(xy))\big)$ for all $n \in \mathbb{Z}$.

Proof. Suppose that all the assumptions in the lemma hold. Therefore, $J_y(xy^m) \neq 0$ for some $m \in \mathbb{Z}$. By the alternatives in $\mathcal{P}f_y^{(2)}(xy^m)$, we get $F_y^{(2)}(xy^m) = 0$. Consider the alternatives in $\mathcal{P}f_y^{(2)}(xy^{m-1})$ as follows: If $F_y^{(2)}(xy^{m-1}) \neq 0$, then Lemma 3.1.4 gives a contradiction. Thus we must have $F_y^{(2)}(xy^{m-1}) = 0$. Similarly, by Lemma 3.1.4, the alternatives in $\mathcal{P}f_y^{(2)}(xy^{m+1})$ gives $F_y^{(2)}(xy^{m+1}) = 0$. First, we will show that $F_y^{(2)}(xy^{m+2}) = 0$ by contradiction. Suppose $F_y^{(2)}(xy^{m+2}) \neq 0$. The alternatives in $\mathcal{P}f_y^{(2)}(xy^{m+2})$ gives $J_y(xy^{m+2}) = 0$. Therefore, $f(xy^{m+2}) \neq 0$ by Lemma 3.1.1. From $F_y^{(2)}(xy^{m+1}) = 0$ and $F_y^{(2)}(xy^{m+2}) \neq 0$, we get $F_y^{(2)}(xy^{m+3}) \neq 0$ by Lemma 3.1.2. Hence the alternatives in $\mathcal{P}f_y^{(2)}(xy^{m+3})$ give $J_y(xy^{m+3}) = 0$. Eliminating $f(xy^m)$ from $F_y^{(2)}(xy^m) = 0$ and $F_y^{(2)}(xy^{m+1}) = 0$, we get

$$f(xy^{m-1}) - 3f(xy^{m+1}) - 2f(xy^{m+2}) = 0. \tag{3.25}$$

Eliminating $f(xy^{m-1})$ and $f(xy^{m+3})$ from $J_y(xy^{m+2}) = 0$, (3.25) and each alternative in $\mathcal{P}f_{y^2}^{(2)}(xy^{m+1})$, we have

$$f(xy^{m+2}) = 0 \text{ or } f(xy^{m+1}) + f(xy^{m+2}) = 0. \tag{3.26}$$

On the other hand, by $J_y(xy^{m+2}) = 0$ and $J_y(xy^{m+3}) = 0$, we obtain that

$$2f(xy^{m+1}) - 3f(xy^{m+2}) + f(xy^{m+4}) = 0. \tag{3.27}$$

Eliminating $f(xy^m)$ and $f(xy^{m+4})$ from $F_y^{(2)}(xy^{m+1}) = 0$, (3.27) and each alternative in $\mathcal{P}f_{y^2}^{(2)}(xy^{m+2})$, we get

$$f(xy^{m+1}) = 0 \text{ or } f(xy^{m+1}) - f(xy^{m+2}) = 0. \tag{3.28}$$

Combining (3.26) and (3.28), we conclude that $f(xy^{m+2}) = 0$, a contradiction. Therefore, we get $F_y^{(2)}(xy^{m+2}) = 0$. We can repeat the above process to show $F_y^{(2)}(xy^{m+3}) = 0$ by substituting x by xy and so on. Thus we obtain that

$$F_y^{(2)}(xy^n) = 0 \text{ for all } n \geq m+2. \tag{3.29}$$

Similarly, we can repeat the process of (3.29) for each $n \leq m-2$ to get $F_y^{(2)}(xy^n) = 0$ for all $n \in \mathbb{Z}$, i.e.,

$$f(xy^{n+1}) + f(xy^n) = (-1)(f(xy^n) + f(xy^{n-1})).$$

Therefore, $f(xy^n) = (-1)^n\big(f(x) - n(f(x) + f(xy))\big)$ for all $n \in \mathbb{Z}$. $\square$

Now we will prove the main theorem.

Theorem 3.2.4. *If there exists a function $f \in \mathcal{A}_{(G,H)}^{(\beta)} \backslash \mathcal{J}_{(G,H)}$, then $\beta \in \{0, 1, 2\}$. Moreover, if $x, y \in G$, then one of the following properties must hold:*

(1) $\beta = 0$ *and* $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{-a}, \overline{a})$ *for some* $a \in H\backslash\{0\}$.

(2) $\beta = 1$ *and*

(2.1) $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{a, b, -a-b})$ *for some* $a, b \in H$ *with* $(a, b) \neq (0, 0)$, *or*

(2.2) $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{a}, -2a, \overline{a})$ *for some* $a \in H\backslash\{0\}$, *or*

(2.3) $\big(f(xy^n)\big)_{n\in\mathbb{Z}} = (\overline{-2a, a, \ldots, a})$, *a periodic sequence of an odd period* $p \geq 5$, *for some* $a \in H\backslash\{0\}$.

(3) $\beta = 2$ *and* $f(xy^n) = (-1)^n\big(f(x) - n(f(x) + f(xy))\big)$ *for all* $n \in \mathbb{Z}$.

Proof. Let $f \in \mathcal{A}^{(\beta)}_{(G,H)}\backslash\mathcal{J}_{(G,H)}$ and let $x, y \in G$. By the above observation, we have the following cases:

(1) Assume that $J_y(xy^n) \neq 0$ for all $n \in \mathbb{Z}$. The alternatives in $\mathcal{P}f^{(\beta)}_y(xy^{-1})$ and $\mathcal{P}f^{(\beta)}_y(xy)$ give $F^{(\beta)}_y(xy^{-1}) = 0$ and $F^{(\beta)}_y(xy) = 0$, respectively. From $F^{(\beta)}_y(xy^{-1}) = 0$, $J_y(x) \neq 0$ and $F^{(\beta)}_y(xy) = 0$, Lemma 3.1.4 gives $\beta = 1$ or $\beta = 2$.

(2) Assume that there exists $m \in \mathbb{Z}$ such that $J_y(xy^m) \neq 0$ and $J_y(xy^{m-1}) = 0$, or $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$. Therefore, each case implies that:

(2.1) If $J_y(xy^m) \neq 0$ and $J_y(xy^{m-1}) = 0$, then Lemma 3.1.2 gives $J_y(xy^{m+1}) \neq 0$. From $J_y(xy^m) \neq 0$ and $\mathcal{P}f^{(\beta)}_y(xy^m)$, we get $F^{(\beta)}_y(xy^m) = 0$. Consider the alternatives in $\mathcal{P}f^{(\beta)}_y(xy^{m+2})$ as follows: If $F^{(\beta)}_y(xy^{m+2}) \neq 0$, then Lemma 3.1.3 gives $\beta = 0$. If $F^{(\beta)}_y(xy^{m+2}) = 0$, then Lemma 3.1.4 gives $\beta = 1$ or $\beta = 2$.

(2.2) If $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$, then we get similar results in the case 2.1 by substituting x by xy^{2m} and y by y^{-1} in the previous arguments.

Thus we must have $\beta \in \{0, 1, 2\}$. According to Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.2.3, we have our results corresponding to the values of β's. $\square$

Corollary 3.2.5. *Let* $f \in \mathcal{A}^{(\beta)}_{(G,H)}$. *If* $\beta \notin \{0, 1, 2\}$, *then* $f \in \mathcal{J}_{(G,H)}$.

Proof. If $\beta \notin \{0, 1, 2\}$, then, from Theorem 3.2.4, $\mathcal{A}^{(\beta)}_{(G,H)}\backslash\mathcal{J}_{(G,H)}$ is empty. Hence we have the desired result. $\square$

In other words, Corollary 3.2.5 states that when $\beta \notin \{0, 1, 2\}$, the alternative Jensen's functional equation (WA) is equivalent to the Jensen's functional equation (1.6) for the class of functions from $(G, \cdot)$ to $(H, +)$. On the other hand, when $\beta \in \{0, 1, 2\}$, (WA) is not necessarily equivalent to (1.6) as the following three examples.

Example 3.2.6. Given $a \in H\backslash\{0\}$. Let $f : \mathbb{R} \to H$ be a function such that

$$f(x) = \begin{cases} -a & \text{if } x < 0, \\ a & \text{if } x \geq 0. \end{cases}$$

By choosing $x = 0$ and $y = 1$, we have

$$f(x-y) - 2f(x) + f(x+y) = f(-1) - 2f(0) + f(1) = -2a.$$

From $a \neq 0$ and H is uniquely divisible, we get $-2a \neq 0$. Thus $f \notin \mathcal{J}_{(\mathbb{R},H)}$. Given $x, y \in \mathbb{R}$, if $x-y \geq 0$ and $x+y \geq 0$, or $x-y < 0$ and $x+y < 0$, then $f(x-y)-2f(x)+f(x+y) = 0$; otherwise, $f(x-y) + f(x+y) = 0$. Therefore, $f \in \mathcal{A}^{(\beta)}_{(R,H)} \backslash \mathcal{J}_{(\mathbb{R},H)}$.

Example 3.2.7. Given $a \in H\backslash\{0\}$. Let $f : \mathbb{R} \to H$ be a function such that

$$f(x) = \begin{cases} -2a & \text{if } x = 0, \\ a & \text{otherwise.} \end{cases}$$

By choosing $x = 0$ and $y = 1$, we obtain that

$$f(x-y) - 2f(x) + f(x+y) = f(-1) - 2f(0) + f(1) = 6a.$$

Since $a \neq 0$ and H is uniquely divisible, we have $6a \neq 0$. Thus $f \notin \mathcal{J}_{(\mathbb{R},H)}$. Given $x, y \in \mathbb{R}$, if $\big(y \neq 0$ and $x - y = 0\big)$ or $\big(y \neq 0$ and $x = 0\big)$ or $\big(y \neq 0$ and $x + y = 0\big)$, then f will satisfy $f(x-y) + f(x) + f(x+y) = 0$; otherwise, we have $f(x-y) - 2f(x) + f(x+y) = 0$. Thus $f \in \mathcal{A}^{(\beta)}_{(\mathbb{R},H)} \backslash \mathcal{J}_{(\mathbb{R},H)}$.

Example 3.2.8. Given $a, b \in H$ with $a \neq -b$. Let $f : \mathbb{Z} \to H$ be a function such that

$$f(n) = (-1)^n (a + nb) \text{ for all } n \in \mathbb{Z}.$$

Note that $f(0) - 2f(1) + f(2) = 4a + 4b$. Since $a \neq -b$ and H is uniquely divisible, we have $4a + 4b \neq 0$. Thus $f \notin \mathcal{J}_{(\mathbb{Z},H)}$. Given $n, m \in \mathbb{Z}$. If m is odd, then we observe that $n - m$ and $n + m$ have the same parity whereas n and $n + m$ have the opposite. Therefore, $f(n - m) + 2f(n) + f(n + m) = 0$. Otherwise, if m is even, then $n - m, n, n + m$ all have the same parity. Hence $f(n - m) - 2f(n) + f(n + m) = 0$. Thus $f \in \mathcal{A}^{(\beta)}_{(\mathbb{Z},H)} \backslash \mathcal{J}_{(\mathbb{Z},H)}$.

CHAPTER IV

THE STRONG FORM OF ALTERNATIVE JENSEN FUNCTIONAL EQUATIONS

In this chapter, we will give a criterion for the existence of the general solution for the functional equation (SA) in chapter I. Let $(G, \cdot)$ be a group, $(H, +)$ be a uniquely divisible abelian group. Given integers α, β, γ as in (1.18) and a function $f : G \to H$. For every pair of $x, y \in G$, we will define

$$F_y^{(\alpha,\beta,\gamma)}(x) := \alpha f(xy^{-1}) + \beta f(x) + \gamma f(xy),$$

and

$$J_y(x) := f(xy^{-1}) - 2f(x) + f(xy).$$

In addition, we denote the statement

$$\mathcal{P}f_y^{(\alpha,\beta,\gamma)}(x) := \Big(J_y(x) = 0 \quad \text{or} \quad F_y^{(\alpha,\beta,\gamma)}(x) = 0 \Big).$$

The set of solution to the statement $\mathcal{P}f_y^{(\alpha,\beta,\gamma)}(x)$ will be denoted by $\mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$, i.e.,

$$\mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)} := \{f : G \to H \mid \mathcal{P}f_y^{(\alpha,\beta,\gamma)}(x) \text{ for all } x, y \in G\},$$

while the set of solution of $J_y(x) = 0$ is denoted by

$$\mathcal{J}_{(G,H)} := \{f : G \to H \mid J_y(x) = 0 \text{ for all } x, y \in G\}.$$

For the sake of convenience, we will refer the conditions of integers α, β and γ as in the following equation,

$$\left.\begin{array}{ll} & \beta = \alpha + \gamma \\ \text{or} & \beta = 0 \text{ and } \gamma = \alpha \\ \text{or} & (\beta, \gamma) = (\alpha, \alpha). \end{array}\right\} \tag{4.1}$$

4.1 Auxiliary Lemmas

Lemma 4.1.1. *Let $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$ and $x, y \in G$.*
If $J_y(x) \neq 0$, then $\alpha = \gamma$ or $f(xy^{-1}) = f(xy)$.

Proof. Assume that $J_y(x) \neq 0$. By the alternative in $\mathcal{P}f_y^{(\alpha,\beta,\gamma)}(x)$ and $\mathcal{P}f_{y^{-1}}^{(\alpha,\beta,\gamma)}(x)$, we get $F_y^{(\alpha,\beta,\gamma)}(x)$ and $F_{y^{-1}}^{(\alpha,\beta,\gamma)}(x)$, respectively. Therefore, $F_y^{(\alpha,\beta,\gamma)}(x) - F_{y^{-1}}^{(\alpha,\beta,\gamma)}(x) = 0$, i.e.,

$$(\alpha - \gamma)(f(xy^{-1}) - f(xy)) = 0.$$

Hence $\alpha = \gamma$ or $f(xy^{-1}) = f(xy)$ as desired. □

In the following lemma, we will give a necessary condition for a function $f \in \mathcal{A}_{(G,H)}^{(0,\beta,0)}$.

Lemma 4.1.2. *If $f \in \mathcal{A}_{(G,H)}^{(0,\beta,0)}$ and $x, y \in G$, then $J_y(x) = 0$.*

Proof. Assume that $f \in \mathcal{A}_{(G,H)}^{(0,\beta,0)}$ and $J_y(x) \neq 0$. If $\beta = 0$, then it is a contradiction to (1.18). Hence we must have $\beta \neq 0$. From $F_y^{(0,\beta,0)}(x) = 0$, we get $f(x) = 0$. Next, we will consider the alternative in $F_y^{(0,\beta,0)}(xy^{-1}) = 0$ as follows.

(1) Assume that $F_y^{(0,\beta,0)}(xy^{-1}) = 0$. We have $f(xy^{-1}) = 0$. By the alternative in $\mathcal{P}f_y^{(0,\beta,0)}(xy)$, we obtain that

$$f(xy^2) - 2f(xy) = 0 \text{ or } f(xy) = 0. \tag{4.2}$$

By the alternative in $\mathcal{P}f_y^{(0,\beta,0)}(xy^2)$, we get

$$f(xy) - 2f(xy^2) + f(xy^3) = 0 \text{ or } f(xy^2) = 0. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain that

$$f(xy^3) - 3f(xy) = 0 \text{ or } f(xy) = 0. \tag{4.4}$$

By $f(xy^{-1}) = 0$ and (4.4), the alternative in $\mathcal{P}f_{y^2}^{(0,\beta,0)}(xy)$ gives $f(xy) = 0$. By calculation, we get $J_y(x) = 0$, a contradiction to the fact that $J_y(x) \neq 0$.

(2) Assume that $J_y(xy^{-1}) = 0$, i.e.,

$$f(xy^{-2}) - 2f(xy^{-1}) = 0. \tag{4.5}$$

By (4.5), the alternative in $\mathcal{P}f_y^{(0,\beta,0)}(xy^{-2})$ gives

$$f(xy^{-3}) - 3f(xy^{-1}) = 0 \text{ or } f(xy^{-1}) = 0. \tag{4.6}$$

By (4.6), the alternative in $\mathcal{P}f_{y^2}^{(0,\beta,0)}(xy^{-1})$ gives

$$f(xy^{-1}) + f(xy) = 0 \text{ or } f(xy^{-1}) = 0.$$

If $f(xy^{-1}) + f(xy) = 0$, then we have $J_y(x) = 0$, a contradiction. Hence we have $f(xy^{-1}) = 0$. By a similar argument in case (1), we get a contradiction.

Therefore, we must have $J_y(x) = 0$ as desired. □

Corollary 4.1.3. *Let $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\alpha)}$ and $x, y \in G$. If $J_y(x) \neq 0$, then $\alpha \neq 0$.*

Proof. The proof is complete by Lemma 4.1.2. □

Lemma 4.1.4. *Let $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$ with $\alpha \neq \gamma$ and $x, y \in G$.*

(1) If $J_y(xy^{-1}) \neq 0$ and $J_y(x) \neq 0$, then $J_y(xy) \neq 0$.

(2) If $J_y(xy^{-1}) = 0$ and $J_y(x) = 0$, then $J_y(xy) = 0$.

Proof. We will prove each property as follows.

(1) Assume that $J_y(xy^{-1}) \neq 0, J_y(x) \neq 0$ but $J_y(xy) = 0$. From $J_y(xy^{-1}) \neq 0$ and $J_y(x) \neq 0$, Lemma 4.1.1 gives

$$f(xy^{-2}) = f(x) \text{ or } f(xy^{-1}) = f(xy). \tag{4.7}$$

Eliminating $f(xy)$ from $J_y(xy) = 0$ and (4.7), we get

$$2f(xy^{-1}) - f(x) - f(xy^2) = 0. \tag{4.8}$$

We will consider the alternative in $\mathcal{P}f_{y^2}^{(\alpha,\beta,\gamma)}(x)$.

(a) Suppose $J_{y^2}(x) \neq 0$. Then Lemma 4.1.1 gives

$$f(x) = f(xy^2). \tag{4.9}$$

From (4.7) and (4.9), we get $J_{y^2}(x) = 0$, a contradiction.

(b) Suppose $J_{y^2}(x) = 0$. Eliminating $f(xy^{-2})$ and $f(xy^2)$ from (4.7), (4.8) and $J_{y^2}(x) = 0$, yields

$$f(xy^{-1}) = f(x). \tag{4.10}$$

From (4.7) and (4.10), we obtain $J_y(x) = 0$, a contradiction.

Thus we must have $J_y(xy) \neq 0$.

(2) Assume that $J_y(xy^{-1}) = 0, J_y(x) = 0$ but $J_y(xy) \neq 0$. Eliminating $f(xy^{-1})$ from $J_y(xy^{-1}) = 0$ and $J_y(x) = 0$, we get

$$f(xy^{-2}) - 3f(x) + 2f(xy) = 0. \tag{4.11}$$

From $J_y(xy) \neq 0$, Lemma 4.1.1 gives

$$f(x) = f(xy^2). \tag{4.12}$$

Next, we will consider the alternative in $\mathcal{P}f_{y^2}^{(\alpha,\beta,\gamma)}(x)$.

(a) Suppose $J_{y^2}(x) \neq 0$. By Lemma 4.1.1, we have

$$f(xy^{-2}) = f(xy^2). \tag{4.13}$$

By (4.12) and (4.13), we obtain that $J_{y^2}(x) = 0$, a contradiction.

(b) Suppose $J_{y^2}(x) = 0$. Eliminating $f(xy^{-2})$ and $f(xy^2)$ from (4.11), (4.12) and $J_{y^2}(x) = 0$, we have

$$f(x) = f(xy). \tag{4.14}$$

By (4.12) and (4.14), we get $J_y(xy) = 0$, a contradiction.

Therefore, we must have $J_y(xy) = 0$.

□

Lemma 4.1.5. *Let $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$ with $\alpha \neq \gamma$ and $x, y \in G$.*

(1) If $J_y(xy^{-1}) \neq 0$ and $J_y(x) = 0$, then $J_y(xy) \neq 0$.

(2) If $J_y(xy^{-1}) = 0$ and $J_y(x) \neq 0$, then $J_y(xy) = 0$.

Proof. We will apply Lemma 4.1.4 to prove this lemma as follows.

(1) Assume that $J_y(xy^{-1}) \neq 0$, $J_y(x) = 0$ and $J_y(xy) = 0$. By $J_y(x) = 0$ and $J_y(xy) = 0$, Lemma 4.1.4 gives $J_y(xy^{-1}) = 0$, a contradiction. Thus we must have $J_y(xy) \neq 0$.

(2) Assume that $J_y(xy^{-1}) = 0, J_y(x) \neq 0$ and $J_y(xy) \neq 0$. From $J_y(x) \neq 0$ and $J_y(xy) \neq 0$, we get $J_y(xy^{-1}) \neq 0$, a contradiction. Therefore, we must have $J_y(xy) = 0$.

□

Lemma 4.1.6. *Let $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$ with $\alpha \neq \gamma$ and $x, y \in G$.*

(1) If $J_y(xy^{-1}) \neq 0$ and $J_y(x) \neq 0$, then $\beta = \alpha + \gamma$.

(2) If $J_y(xy^{-1}) = 0$ and $J_y(x) \neq 0$, then $(\beta, \gamma) = (0, -\alpha)$.

Proof. We will prove each property as follows.

(1) Assume that $J_y(xy^{-1}) \neq 0$, $J_y(x) \neq 0$ and $\beta \neq \alpha + \gamma$. By Lemma 4.1.1, we obtain that

$$f(xy^{-2}) = f(x) \text{ and } f(xy^{-1}) = f(xy). \tag{4.15}$$

From $J_y(xy^{-1}) \neq 0$ and $J_y(x) \neq 0$, the alternative in $\mathcal{P}f_y^{(\alpha,\beta,\gamma)}(xy^{-1})$ and $\mathcal{P}f_y^{(\alpha,\beta,\gamma)}(x)$ gives

$$F_y^{(\alpha,\beta,\gamma)}(xy^{-1}) = 0 \text{ and } F_y^{(\alpha,\beta,\gamma)}(x) = 0, \tag{4.16}$$

respectively. By (4.15) and (4.16), we get

$$(\alpha + \gamma)f(x) + \beta f(xy) = 0 \text{ and } \beta f(x) + (\alpha + \gamma)f(xy) = 0. \tag{4.17}$$

Eliminating $f(xy)$ from (4.9), we have

$$(\beta - \alpha - \gamma)(\beta + \alpha + \gamma)f(x) = 0.$$

From $\beta \neq \alpha + \gamma$, we can conclude that $f(x) = 0$ or $\beta = -\alpha - \gamma$.

(a) Suppose $f(x) = 0$. By (4.17), we have

$$\beta f(xy) = 0 \text{ and } (\alpha + \gamma)f(xy) = 0.$$

If $f(xy) = 0$, then $f(xy^{-1}) = 0$ by (4.15) and we can calculate $J_y(x) = 0$, a contradiction to the fact that $J_y(x) \neq 0$. Thus we get $(\alpha + \gamma) = 0$ and $\beta = 0$, a contradiction to the fact that $\beta \neq \alpha + \gamma$.

(b) Suppose $\beta = -\alpha - \gamma$. Substituting $\beta = -\alpha - \gamma$ in (4.17), we obtain that

$$(\alpha + \gamma)(f(x) - f(xy)) = 0.$$

If $\alpha + \gamma = 0$, then $\beta = 0$ which contradicts $\beta \neq \alpha + \gamma$. Thus we must have $f(x) = f(xy)$. Since $f(xy^{-1}) = f(xy)$ in (4.15), we get $J_y(x) = 0$, a contradiction to the fact that $J_y(x) \neq 0$.

Hence $\beta = \alpha + \gamma$.

(2) Assume that $J_y(xy^{-1}) = 0$ and $J_y(x) \neq 0$. From $J_y(x) \neq 0$, by Lemma 4.1.1, we have

$$f(xy^{-1}) = f(xy). \tag{4.18}$$

By $J_y(x) \neq 0$ again, the alternative in $\mathcal{P}f_y^{(\alpha,\beta,\gamma)}(x)$ gives $F_y^{(\alpha,\beta,\gamma)}(x) = 0$. Substituting $f(xy^{-1})$ from (4.18) in $F_y^{(\alpha,\beta,\gamma)}(x) = 0$, we obtain that

$$\beta f(x) + (\alpha + \gamma) f(xy) = 0. \tag{4.19}$$

On the other hand, we substituting $f(xy^{-1})$ from (4.18) in $J_y(xy^{-1}) = 0$ to get

$$f(xy^{-2}) - 2f(xy) + f(x) = 0. \tag{4.20}$$

Since $J_y(xy^{-1}) = 0$ and $J_y(x) \neq 0$, Lemma 4.1.5 give $J_y(xy^{-2}) \neq 0$. By Lemma 4.1.1 and (4.18), we have

$$f(xy^{-3}) = f(xy). \tag{4.21}$$

From $J_y(xy^{-2}) \neq 0$, the alternative in $\mathcal{P}f_y^{(\alpha,\beta,\gamma)}(xy^{-2})$ gives $F_y^{(\alpha,\beta,\gamma)}(xy^{-2}) = 0$. By (4.18), (4.21) and $F_y^{(\alpha,\beta,\gamma)}(xy^{-2}) = 0$, we get

$$(\alpha + 2\beta + \gamma) f(xy) - \beta f(x) = 0. \tag{4.22}$$

By (4.19), (4.22) and simplifying, we obtain that

$$\beta(f(x) - f(xy)) = 0.$$

If $f(x) - f(xy) = 0$, then by (4.18), we have $J_y(x) = 0$, a contradiction to the fact that $J_y(x) \neq 0$. Hence we must get $\beta = 0$. Thus (4.22) gives

$$(\alpha + \gamma) f(xy) = 0. \tag{4.23}$$

Suppose in a contrary that $\alpha + \gamma \neq 0$. Thus $f(xy) = 0$ and so is $f(xy^{-1})$ by (4.18). By (4.20), we obtain that

$$f(xy^{-2}) + f(x) = 0. \tag{4.24}$$

From $J_y(xy^{-1}) = 0$ and $J_y(x) \neq 0$, Lemma 4.1.5 give $J_y(xy) = 0$, i.e.,

$$f(x) + f(xy^2) = 0. \tag{4.25}$$

Thus by (4.24), (4.25) and $\alpha + \gamma \neq 0$, the alternative in $\mathcal{P}f_{y^2}^{(\alpha,0,\gamma)}(x)$ give $f(x) = 0$. Then $J_y(x) = 0$, a contradiction. Therefore, we must get $\alpha + \gamma = 0$ and so $(\alpha, \beta, \gamma) = (\alpha, 0, -\alpha)$.

□

4.2 Main Results and Some Examples

Before proving the theorem, we will provide the following two lemmas which will eventually be used in our main theorem.

Lemma 4.2.1. *If $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\alpha)} \backslash \mathcal{J}_{(G,H)}$ and $x, y \in G$, then one of the following properties holds:*

(1) $\beta = 0$ *and* $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{-a, \overline{a}})$ *for some* $a \in H$.

(2) $\beta = \alpha$ *and*

(2.1) $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{a, b, -a-b})$ *for some* $a, b \in H$*, or*

(2.2) $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{a}, -2a, \overline{a})$ *for some* $a \in H$*, or*

(2.3) $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{-2a, a, \ldots, a})$*, a periodic sequence of an odd period* $p \geq 5$*, for some* $a \in H$.

(3) $\beta = 2\alpha$ *and* $f(xy^n) = (-1)^n\big(f(x) - n(f(x) + f(xy))\big)$ *for all* $n \in \mathbb{Z}$.

Proof. Assume that the assumption in the lemma holds. From Corollary 4.1.3, we must have $\alpha \neq 0$. By direct substitution, we get $\mathcal{A}_{(G,H)}^{(\alpha,\beta,\alpha)} \backslash \mathcal{J}_{(G,H)} = \mathcal{A}_{(G,H)}^{(1,\frac{\beta}{\alpha},1)} \backslash \mathcal{J}_{(G,H)}$. Thus Theorem 3.2.4 gives $\beta \in \{0, \alpha, 2\alpha\}$, and f satisfies one of the properties in the lemma. $\square$

Lemma 4.2.2. *If* $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)} \backslash \mathcal{J}_{(G,H)}$ *with* $\alpha \neq \gamma$ *and* $x, y \in G$*, then* $\beta = \alpha + \gamma$ *and one of the following properties holds:*

(1) $f(xy^n) = (-1)^n a$ *for all* $n \in \mathbb{Z}$ *and for some* $a \in H$*, or*

(2) $\beta = 0$ *and*

(2.1) $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{a, b})$ *for some* $a, b \in H$*, or*

(2.2) $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{2a - b, a, b, a})$ *for some* $a, b \in H$.

Proof. Assume that the assumption in the lemma holds. By the definition of $\mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$ and $\mathcal{J}_{(G,H)}$, one of the following properties holds:

(1) $J_y(xy^n) \neq 0$ for all $n \in \mathbb{Z}$.

(2) There exists $m \in \mathbb{Z}$ such that

(2.1) $J_y(xy^m) \neq 0$ and $J_y(xy^{m-1}) = 0$, or

(2.2) $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$.

(1) Assume that $J_y(xy^n) \neq 0$ for all $n \in \mathbb{Z}$. Lemma 4.1.6 gives $\beta = \alpha + \gamma$. By Lemma 4.1.1, we get

$$f(xy^{n-1}) = f(xy^{n+1}) \text{ for all } n \in \mathbb{Z}. \tag{4.26}$$

From $J_y(x) \neq 0$, the alternative in $\mathcal{P}f_y^{(\alpha,\alpha+\gamma,\gamma)}(x)$ gives $F_y^{(\alpha,\alpha+\gamma,\gamma)}(x) = 0$; that is,

$$\alpha f(xy^{-1}) + (\alpha + \gamma)f(x) + \gamma f(xy) = 0. \tag{4.27}$$

By (4.26) with $n = 0$ and (4.27), we get

$$(\alpha+\gamma)(f(x)+f(xy)) = 0. \tag{4.28}$$

(1.1) Suppose $f(x)+f(xy) = 0$. Let $f(x) = a$. We have $f(xy) = -a$. By (4.26), we conclude that

$$f(xy^n) = \begin{cases} a & \text{if } n \text{ is even,} \\ -a & \text{if } n \text{ is odd.} \end{cases}$$

Hence we get $f(xy^n) = (-1)^n a$ for all $n \in \mathbb{Z}$.

(1.2) Suppose $\alpha+\gamma = 0$. That is $\beta = 0$. Let $f(x) = a$ and $f(xy) = b$. Therefore, by (4.26), we obtain that

$$f(xy^n) = \begin{cases} a & \text{if } n \text{ is even,} \\ b & \text{if } n \text{ is odd.} \end{cases}$$

Thus we have $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{a, b})$.

(2) Assume that there exists $m \in \mathbb{Z}$ such that $J_y(xy^m) \neq 0$ and $J_y(xy^{m-1}) = 0$ or $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$. Thus Lemma 4.1.6 gives $(\beta, \gamma) = (0, -\alpha)$

(2.1) Suppose $J_y(xy^m) \neq 0$ and $J_y(xy^{m-1}) = 0$. Let $f(xy^{m-1}) = a$ and $f(xy^m) = b$. From $J_y(xy^m) \neq 0$, Lemma 4.1.1 gives $f(xy^{m-1}) = f(xy^{m+1})$, i.e., $f(xy^{m+1}) = a$. From $J_y(xy^{m-1}) = 0$, we get $f(xy^{m-2}) = 2a - b$. Now we have

$$(f(xy^{m-2}), f(xy^{m-1}), f(xy^m), f(xy^{m+1})) = (2a-b, a, b, a). \tag{4.29}$$

From $J_y(xy^{m-1}) = 0$ and $J_y(xy^m) \neq 0$, Lemma 4.1.5 gives $J_y(xy^{m+1}) = 0$; that is,

$$f(xy^m) - 2f(xy^{m+1}) + f(xy^{m+2}) = 0. \tag{4.30}$$

By (4.29) and (4.30), we obtain that $f(xy^{m+2}) = 2a - b$. Since $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$, Lemma 4.1.5 gives $J_y(xy^{m+2}) \neq 0$. By Lemma 4.1.1, we get $f(xy^{m+3}) = a$. From $J_y(xy^{m+1}) = 0$ and $J_y(xy^{m+2}) \neq 0$, Lemma 4.1.5 gives $J_y(xy^{m+3}) = 0$ and so $f(xy^{m+4}) = b$. As $J_y(xy^{m+2}) \neq 0$ and $J_y(xy^{m+3}) = 0$, we have $J_y(xy^{m+4}) \neq 0$ by Lemma 4.1.5. Lemma 4.1.1 gives $f(xy^{m+5}) = a$. Thus we obtain that

$$(f(xy^{m+2}), f(xy^{m+3}), f(xy^{m+4}), f(xy^{m+5})) = (2a - b, a, b, a). \tag{4.31}$$

Similarly, by repeating the process of (4.31), we get

$$(f(xy^{m+6}), f(xy^{m+7}), f(xy^{m+8}), f(xy^{m+9})) = (2a - b, a, b, a)$$

and so on. Eventually, we arrive that

$$\begin{aligned}(f(xy^{m-2+4i}), f(xy^{m-1+4i}), f(xy^{m+4i}), &f(xy^{m+1+4i})) \\ &= (2a - b, a, b, a)\end{aligned} \tag{4.32}$$

for all $i \geq 0$. Moreover, we can similarly repeat the process of (4.32) for each $f(xy^k)$ with $k \leq m - 3$ to get $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{2a - b, a, b, a})$.

(2.2) Suppose $J_y(xy^m) \neq 0$ and $J_y(xy^{m+1}) = 0$. By substituting x by xy^{2m} and y by y^{-1} in the arguments in the case (2.1), we have similar results.

□

Now we are ready to prove the main theorem.

Theorem 4.2.3. *If $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)} \backslash \mathcal{J}_{(G,H)}$, then (4.1) holds. Moreover, if $x, y \in G$, then one of the following properties holds:*

(1) $\beta = \alpha + \gamma$ and

(1.1) $f(xy^n) = (-1)^n a$ *for all* $n \in \mathbb{Z}$ *and for some* $a \in H$*, or*

(1.2) $\beta = 0$ *and*

(1.2.1) $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{a, b})$ *for some* $a, b \in H$*, or*

(1.2.2) $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{2a - b, a, b, a})$ *for some* $a, b \in H$*, or*

(1.3) $\beta = 2\alpha$ *and* $f(xy^n) = (-1)^n\big(f(x) - n(f(x) + f(xy))\big)$ *for all* $n \in \mathbb{Z}$*.*

(2) $(\beta, \gamma) = (0, \alpha)$ *and* $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{-a}, \overline{a})$ *for some* $a \in H$*.*

(3) $(\beta, \gamma) = (\alpha, \alpha)$ *and*

(3.1) $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{a, b, -a-b})$ *for some* $a, b \in H$*, or*

(3.2) $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{a}, -2a, \overline{a})$ *for some* $a \in H$*, or*

(3.3) $(f(xy^n))_{n\in\mathbb{Z}} = (\overline{-2a, a, \ldots, a})$*, a periodic sequence of an odd period* $p \geq 5$*, for some* $a \in H$*.*

Proof. Assume that all assumptions in the theorem hold. We will consider the case of an integer α as follows:

(1) If $\alpha = \gamma$, then Lemma 4.2.1 gives $\beta \in \{0, \alpha, 2\alpha\}$ and the properties (1.3), (2) and (3).

(2) If $\alpha \neq \gamma$, then Lemma 4.2.2 gives $\beta = \alpha + \gamma$ and the properties (1.1) and (1.2).

□

Corollary 4.2.4. *Let* $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$*. If (4.1) does not hold, then* $f \in \mathcal{J}_{(G,H)}$*.*

Proof. If (4.1) does not hold, then, from Theorem 4.2.3, $\mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)} \backslash \mathcal{J}_{(G,H)}$ is empty. Hence we have the desired result. □

In other words, Corollary 4.2.4 states that when (4.1) does not hold, the alternative Jensen's functional equation (SA) is equivalent to the Jensen's functional equation (1.6) for the class of functions from $(G, \cdot)$ to $(H, +)$. On the other hand, when (4.1) actually holds, (SA) in chapter I is not necessarily equivalent to (1.6). As in Example 3.2.7, we also get $f \in \mathcal{A}_{(\mathbb{Z},H)}^{(\alpha,\alpha,\alpha)} \backslash \mathcal{J}_{(\mathbb{Z},H)}$. We will give the following two more examples when (SA) is not necessarily equivalent to (1.6) with $\beta = \alpha + \gamma$ or $(\beta, \gamma) = (0, \alpha)$.

Example 4.2.5. Given $a \in H \backslash \{0\}$. Let $f : \mathbb{Z} \to H$ be a function such that

$$f(n) = (-1)^n a \text{ for all } n \in \mathbb{Z}.$$

Note that

$$f(0) - 2f(1) + f(2) = 4a.$$

From $a \neq 0$ and H is uniquely divisible, we get $4a \neq 0$. Thus $f \notin \mathcal{J}_{(\mathbb{Z},H)}$. Given $n, m \in \mathbb{Z}$. If m is odd, then we observe that $n - m$ and $n + m$ have the same parity whereas n and $n + m$ have the opposite. Therefore,

$$\alpha f(n-m) + (\alpha + \gamma) f(n) + \gamma f(n+m) = 0.$$

Otherwise, if m is even, then $n - m, n, n + m$ all have the same parity. Hence

$$f(n-m) - 2f(n) + f(n+m) = 0.$$

Therefore, $f \in \mathcal{A}_{(\mathbb{Z},H)}^{(\alpha,\alpha+\gamma,\gamma)} \backslash \mathcal{J}_{(\mathbb{Z},H)}$.

Example 4.2.6. Given $a, b \in H$ with $a \neq b$. Let $f : \mathbb{Z} \to H$ be a function such that

$$f(n) = \begin{cases} a & \text{if } n \text{ is even,} \\ b & \text{if } n \text{ is odd.} \end{cases}$$

Note that

$$f(0) - 2f(1) + f(2) = 2a - 2b.$$

Since $a \neq b$ and H is uniquely divisible, we have $2a - 2b \neq 0$. Thus $f \notin \mathcal{J}_{(\mathbb{Z},H)}$. If m is odd, then we observe that $n-m$ and $n+m$ have the same parity. Therefore,

$$\alpha f(n-m)-\alpha f(n+m)=0.$$

Otherwise, if m is even, then $n-m, n, n+m$ all have the same parity. Hence

$$f(n-m)-2f(n)+f(n+m)=0.$$

Thus $f \in \mathcal{A}_{(\mathbb{Z},H)}^{(\alpha,0,-\alpha)} \backslash \mathcal{J}_{(\mathbb{Z},H)}$.

4.3 General Solution on cyclic groups

In this section, we will give the general solution of the alternative Jensen functional equation (SA) in chapter I on an infinite cyclic group and a finite cyclic group. There are mainly the applications of Theorem 4.2.3.

First, we will find all solutions of an infinite cyclic group as in the following theorem.

Theorem 4.3.1. *Let $(G,\cdot)$ be an infinite cyclic group with $G=\langle g\rangle$.*
$f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$ if and only if $f \in \mathcal{J}_{(G,H)}$ or one of the following properties must hold:

(1) $\beta=\alpha+\gamma$ and

(1.1) $f(g^n)=(-1)^n a$ for all $n \in \mathbb{Z}$ and for some $a \in H$, or

(1.2) $\beta=0$ and

(1.2.1) $(f(g^n))_{n\in\mathbb{Z}}=(\overline{a,b})$ for some $a,b \in H$, or

(1.2.2) $(f(g^n))_{n\in\mathbb{Z}}=(\overline{2a-b,a,b,a})$ for some $a,b \in H$, or

(1.3) $\beta=2\alpha$ and $f(g^n)=(-1)^n\big(a+nb\big)$ for all $n \in \mathbb{Z}$ and for some $a,b \in H$.

(2) $(\beta,\gamma)=(0,\alpha)$ and $(f(g^n))_{n\in\mathbb{Z}}=(\overline{-a,a})$ for some $a \in H$.

(3) $(\beta,\gamma)=(\alpha,\alpha)$ and

(3.1) $(f(g^n))_{n\in\mathbb{Z}} = (\overline{a, b, -a-b})$ *for some* $a, b \in H$*, or*

(3.2) $(f(g^n))_{n\in\mathbb{Z}} = (\overline{a}, -2a, \overline{a})$ *for some* $a \in H$*, or*

(3.3) $(f(g^n))_{n\in\mathbb{Z}} = (\overline{-2a, a, \ldots, a})$*, a periodic sequence of an odd period* $p \geq 5$*, for some* $a \in H$.

Proof. Assume that $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$. If $f \notin \mathcal{J}_{(G,H)}$, then setting $x = e$ and $y = g$ in Theorem 4.2.3, we get that one of the properties (1), (2), and (3) must hold. The converse can be directly verified. $\square$

Next, we will give the general solution of a finite cyclic group as in the following theorem.

Theorem 4.3.2. *Let* $(G, \cdot)$ *be a finite cyclic group of order* $m \geq 2$ *with* $G = \langle g \rangle$.
$f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$ *if and only if* $f \in \mathcal{J}_{(G,H)}$ *or one of the following properties must hold:*

(1) $\beta = \alpha + \gamma$,

(1.1) $2 \mid m$ *and* $f(g^n) = (-1)^n a$ *for all* $n \in \mathbb{Z}$ *and for some* $a \in H$*, or*

(1.2) $\beta = 0$,

(1.2.1) $2 \mid m$ *and* $(f(g^n))_{n\in\mathbb{Z}} = (\overline{a, b})$ *for some* $a, b \in H$*, or*

(1.2.2) $4 \mid m$ *and* $(f(g^n))_{n\in\mathbb{Z}} = (\overline{2a-b, a, b, a})$ *for some* $a, b \in H$*, or*

(2) $(\beta, \gamma) = (\alpha, \alpha)$,

(3.1) $3 \mid m$ *and* $(f(g^n))_{n\in\mathbb{Z}} = (\overline{a, b, -a-b})$ *for some* $a, b \in H$*, or*

(3.2) $(f(g^n))_{n\in\mathbb{Z}} = (\overline{-2a, a, \ldots, a})$*, a periodic sequence of an odd period* $p \geq 5$ *with* $p \mid m$*, for some* $a \in H$.

Proof. Given one of the above properties, we can directly verify that $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)}$. Conversely, assume that $f \in \mathcal{A}_{(G,H)}^{(\alpha,\beta,\gamma)} \backslash \mathcal{J}_{(G,H)}$. By setting $x = e$ and $y = g$ in Theorem 4.2.3, we have the possibilities in Theorem 4.3.1.

However, all the above possibilities are not admissible. Some cases are redundant and some cases are admissible with some additional conditions.

(1) Assume that $\beta = \alpha + \gamma$.

(1.1) Suppose that $f(g^n) = (-1)^n a$ for all $n \in \mathbb{Z}$, for some $a \in H$ and m is odd. We get

$$a = f(e) \text{ and } f(g^m) = -a.$$

Since $g^m = e$, therefore $a = 0$ and so $f \in \mathcal{J}_{(G,H)}$, a contradiction. Thus m must be even.

(1.2) Suppose that $(f(g^n))_{n\in\mathbb{Z}} = (\overline{a, b})$ for some $a, b \in H$ and m is odd. Without loss of generality, we let $f(e) = a$. Then $f(g^m) = b$. Since $g^m = e$, $a = b$ and thus $f \in \mathcal{J}_{(G,H)}$, a contradiction. Hence m must be even.

(1.3) Suppose that $(f(g^n))_{n\in\mathbb{Z}} = (\overline{2a - b, a, b, a})$ for some $a, b \in H$ and $4 \nmid m$. Then there exists $k \in \mathbb{Z}$ such that $f(g^k) = 2a - b$. Since $4 \nmid m$, $f(g^{k+m}) \in \{a, b\}$. From m is the order of the group G, we have $g^k = g^{k+m}$. Hence

$$2a - b = a \text{ or } 2a - b = b,$$

which gives $a = b$. Thus $f \in \mathcal{J}_{(G,H)}$, a contradiction. Therefore, we must get $4 \mid m$.

(1.4) Suppose that $f(g^n) = (-1)^n\big(a + nb\big)$ for all $n \in \mathbb{Z}$ and for some $a, b \in H$. Since $e = g^m = g^{2m}$,

$$a = (-1)^m(a + mb) = (-1)^{2m}(a + 2mb)$$

which implies that $b = 0$ and m is even.

(2) Assume that $(\beta, \gamma) = (0, \alpha)$ and $(f(g^n))_{n\in\mathbb{Z}} = (\overline{-a}, \overline{a})$ for some $a \in H$. Thus there exists $k \in \mathbb{Z}$ such that

$$f(g^n) = \begin{cases} -a & \text{if } n < k, \\ a & \text{if } n \geq k. \end{cases}$$

Hence $f(g^{k-1}) = -a$ and $f(g^{k+m-1}) = a$. Since m is the order of the group G, we have

$g^{k-1} = g^{k+m-1}$. Thus we must get $a = 0$ and so $f \in \mathcal{J}_{(G,H)}$, a contradiction. Therefore, this case will not occur.

(3) Assume that $(\beta, \gamma) = (\alpha, \alpha)$.

(3.1) Suppose that $(f(g^n))_{n\in\mathbb{Z}} = (\overline{a, b, -a-b})$ for some $a, b \in H$ and $3 \nmid m$. Then $\{0, m, 2m\}$ is a complete residue modulo 3. Therefore

$$\{f(e), f(g^m), f(g^{2m})\} = \{a, b, -a-b\}.$$

Since m is the order of G, thus $g^{2m} = g^m = e$. Therefore, $a = b = -a-b$, which gives $a = b = 0$ and, in turn, $f \in \mathcal{J}_{(G,H)}$, a contradiction. Hence $3 \mid m$.

(3.2) $(f(g^n))_{n\in\mathbb{Z}} = (\overline{a}, -2a, \overline{a})$ for some $a \in H$. Then there exists $k \in \mathbb{Z}$ such that

$$f(g^n) = \begin{cases} -2a & \text{if } n = k, \\ a & \text{otherwise.} \end{cases}$$

Hence $f(g^k) = -2a$ and $f(g^{k+m}) = a$. Since m is the order of the group G, we have $g^k = g^{k+m}$. Thus we must have $a = 0$ and so $f \in \mathcal{J}_{(G,H)}$, a contradiction. Therefore, this case does not occur.

(3.3) Suppose that $(f(g^n))_{n\in\mathbb{Z}} = (\overline{-2a, a, \ldots, a})$, a periodic sequence of an odd period $p \geq 5$, for some $a \in H$ and $p \nmid m$. Thus there is $k \in \mathbb{Z}$ such that

$$f(g^k) = -2a.$$

Since $\big(f(g^n)\big)_{n\in\mathbb{Z}}$ is periodic sequence of a period p with $p \nmid m$, we must have $f(g^{k+m}) = a$. But m is the order of G, thus $g^{k+m} = g^k$. Therefore, $-2a = a$, which gives $a = 0$, and, in turn, $f \in \mathcal{J}_{(G,H)}$, a contradiction. Hence $p \mid m$.

By all of the above considerations, we are done. $\square$

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