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NONEQUILIBRIUM QUANTUM FIELD THEORY IN COSMOLOGY

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ทฤษฎีสนามควอนตัมในสภาวะไม่สมดุลเป็นเครื่องมือสำคัญในการศึกษาปรากฏการณ์ทางฟิสิกส์ในกลุ่มฟิสิกส์อนุภาคและจักรวาลวิทยา จุดเด่นของทฤษฎีสนามควอนตัมในสภาวะไม่สมดุลคือรูปนัยนิยมที่พิจารณาสนามเฉลี่ยกับตัวแผ่กระจายเป็นตัวแปรอิสระ

ในวิทยานิพนธ์นี้ได้ทำการทบทวนทฤษฎีสนามควอนตัมในสภาวะไม่สมดุลซึ่งประกอบไปด้วยอินทิกรัลบนวิถีปิด และรูปนัยนิยมที่ลดทอนไม่ได้ด้วยสองอนุภาค จากนั้นได้นำกระบวนการของทฤษฎีสนามควอนตัมในสภาวะไม่สมดุลไปใช้ในการศึกษาพลวัตของอนุภาคอินเฟลตอนในปริภูมิแบบโค้ง ทั้งกรณีที่มีและไม่มีการคู่ควบกับสนามของเฟอร์มิออน และได้ทำการอนุพัทธ์สมการพลวัตของสนามเฉลี่ยและตัวแผ่กระจายในทั้งสองกรณี ซึ่งสมการพลวัตที่ได้มีคุณสมบัติของความเป็นเหตุเป็นผล และมีศักยภาพที่จะทำนายพฤติกรรมการสั่นแบบมีความหน่วงของอนุภาคอินเฟลตอน และการผลิตอนุภาคเฟอร์มิออน ถึงแม้ว่าจะไม่สามารถหาผลเฉลยเชิงวิเคราะห์ได้

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Nonequilibrium quantum field theory has recently become an important tool in research in particle physics and cosmology. The salient feature of nonequilibrium quantum field theory is its formalism which treats the propagators as independent variables.

In this thesis, the formulation of nonequilibrium quantum field theory, consisting of the closed-time path integral and the 2-particle-irreducible formalism, is reviewed. The methods are then applied to the study of inflaton dynamics in curved spacetime, with and without fermion coupling, and the dynamical equations for the inflaton mean field and propagators are derived. It is found that these equations respect causality and have the potential of predicting the inflaton damping and the fermion production, even though their solution cannot be obtained analytically.

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CHAPTER I

INTRODUCTION

The formalism of quantum field theory familiar to particle physicists is normally restricted to the description of equilibrium systems, at both zero and finite temperatures. During the past several years, many physicists have become interested in the development of nonequilibrium quantum field theory since there are many quantum systems with nonequilibrium behaviors, whose mathematical description cannot be obtained by using the equilibrium formalism.

The formulation of nonequilibrium quantum field theory is based on the formalism developed by Schwinger and Keldysh, in which the time dimension has a form of a closed loop, going from the infinite past to the infinite future and then going back to the infinite past again. The corresponding path integral formalism is know as the "closed-time path integral" [1, 2, 3]. When combined with the "2-particle-irreducible" formalism [4, 5], the closed-time path integration provides a powerful tool for studying the nonequilibrium quantum systems [6].

An important application of the nonequilibrium quantum field theory is in the area of inflationary cosmology, in which a scalar field, called an inflaton, plays an important role. As the temperature of the early universe is very high, we have no right to assume that the physical processes of the early universe occurred in the equilibrium situations. Thus the dynamics of the inflaton during and after the inflation era has to be treated as the nonequilibrium dynamics. In Ref. [7], the inflaton dynamics in the flat Friedmann-Robertson-Walker spacetime was investigated where the damping of the inflaton field has been found. The effects of fermion coupling to the inflaton have been studied in Ref. [8], where the authors discussed the possibility of having fermion production after the end of inflation.

The purpose of this thesis is to review the methods and results of the nonequilibrium quantum field theory as applied to cosmology. The organization of this thesis is as follows. We introduce the closed-time path integrals and the 2-particle-irreducible formalism in Chapter 2. In Chapter 3, after a brief review on inflationary cosmology, we derive the dynamical equations of the nonequilibrium scalar field theory in curved spacetime and consider a specific case of the Friedmann-Robertson-Walker metric. We then go on to obtain the dynamical equations for the scalar field coupled to a fermion field in Chapter 4, and calculate the nonlocal kernel that appears in the dynamical equation for the propagator. Finally, the conclusions are made in Chapter 5.

CHAPTER II

CLOSED-TIME PATH INTEGRALS AND 2-PARTICLE IRREDUCIBLE FORMALISM

In this chapter, we will discuss the closed-time path integral technique and the 2-particle irreducible formalism, which are the main tools for attacking problems in nonequilibrium quantum field theory.

2.1 Closed-Time Path Integrals

Before discussing the closed-time path integral formalism, we first recall that an important object in the conventional quantum field theory is the generating functional (or the vacuum-to-vacuum transition amplitude) in the presence of an external source J, defined by

$$Z[J] = \langle 0(+\infty)|U_J(+\infty,-\infty)|0(-\infty)\rangle, \tag{2.1}$$

where $U(+\infty,-\infty)$ is the time-evolution operator linking the vacuum states in the infinite past and in the infinite future. For a scalar field theory, $U(+\infty,-\infty)$ takes the form

$$U_J(+\infty,-\infty) = T\left(\exp\frac{i}{\hbar}(H+\Phi J)\right), \tag{2.2}$$

where H and Φ are the Hamiltonian and the scalar field, respectively, and T is the time-ordering operator. In the above expression, an integral over d^4x of the exponent is understood without explicitly writing it; we shall use this convention from now on. It is important to note that the vacuum states in Eq. (2.1) are the ground states at different time, and it is possible that these ground states are different when the system under consideration undergoes a phase transition or evolves in a nonequilibrium situation. Using the generating functional, the connected generating functional $W[J]$ is defined by

$$W[J] = -i\hbar \ln Z[J]. \tag{2.3}$$

The effective action $\Gamma[\phi]$ is defined as the Legendre transform of $W[J]$,

$$\Gamma[\phi] = W[J] - \phi J, \tag{2.4}$$

where $\phi(x) \equiv \delta W[J]/\delta J(x)$ is called the "classical field" or the "mean field" corresponding to the quantum field $\Phi(x)$ [9]. Using this definition of $\phi(x)$, one can see that if the vacua in the asymptotic past and in the far future are equivalent in the sense that the vacuum state in the asymptotic past evolves uniquely into the vacuum state in the far future, $\phi(x)$ is just the vacuum expectation value of the field operator $\Phi(x)$ at time $t = x^0$, and the value of $\phi(x)$ can be obtained by solving the equation

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = 0. \tag{2.5}$$

However, in the nonequilibrium situation in which things change with time, all we know are the initial conditions of the system, and so we cannot be sure if the vacuum state in the infinite past will evolve uniquely into the vacuum state in the far future. Having no idea of what the vacuum state in the far future looks like, it is impossible to calculate the generating functional defined in Eq. (2.1) and so the above formalism fails. To overcome this difficulty, one introduces the closed-time path integral (CTP) formalism, in which Eq. (2.1) is replaced by the "closed-time path generating functional" [2, 3, 10]

$$Z[J_1, J_2] = \langle 0(-\infty)|U_{J_2}(-\infty, +\infty)U_{J_1}(+\infty, -\infty)|0(-\infty)\rangle \tag{2.6}$$

where $U_{J_1}(+\infty, -\infty)$ is defined in Eq. (2.2) with external source J_1

$$U_{J_1}(+\infty, -\infty) = T\left(\exp\frac{i}{\hbar}(H_1 + \Phi_1 J_1)\right), \tag{2.7}$$

and $U_{J_2}(-\infty,+\infty)$ is defined with the anti-temporal ordering operator $\tilde{T}$ instead of the time-ordering one

$$U_{J_2}(+\infty,-\infty)=\tilde{T}\left(\exp\frac{i}{\hbar}(H_2+\Phi_2 J_2)\right) . \qquad (2.8)$$

In this formalism, there are two fields Φ_1 and Φ_2, which evolve forward in time with source J_1 and backward in time with source J_2, respectively. Actually, these two fields correspond to the same field Φ of the original theory; we just use the indices 1 and 2 to distinguish between the systems that evolve in the forward and in the backward time directions. The Hamiltonians H_1 and H_2 are of the same form, except that H_1 and H_2 are, respectively, the functionals of Φ_1 and Φ_2. The interpretation of this CTP generating functional is as follows. Starting with the ground state $|0(-\infty)\rangle$ in the asymptotic past, we evolve it using $U_{J_1}(+\infty,-\infty)$ (with the source J_1) into the infinite future, and then bring it back to the infinite past using $U_{J_2}(-\infty,+\infty)$ (with the source J_2), where it becomes the initial ground state $|0(-\infty)\rangle$ again. We thus see that this formalism does not require any knowledge about things in the future; all we need to know are the initial conditions at $t=-\infty$.

Using the CTP generating functional, the connected generating functional $W[J_1,J_2]$ and the effective action $\Gamma[\phi_1,\phi_2]$ are defined by

$$W[J_1,J_2]=-i\hbar\ln Z[J_1,J_2], \qquad (2.9)$$

and

$$\Gamma[\phi_1,\phi_2]=W[J_1,J_2]-(\phi_1 J_1-\phi_2 J_2), \qquad (2.10)$$

where $\phi_a(x)\equiv(-1)^{a-1}\delta W[J_1,J_2]/\delta J_a(x)$[1] $(a=1,2)$ and the minus sign in front of the last term came from the fact that the second half of the time development

[1]The factor $(-1)^{a-1}$ must be added because the exponent in the definition of U_{J_2} contains a factor $\int_{\infty}^{-\infty}dt\,\Phi_2 J_2=-\int_{-\infty}^{\infty}dt\,\Phi_2 J_2$.

is anti-temporal while the integral representing the term $\phi_2 J_2$ is along the forward time direction. To find the meaning of $\phi_a(x)$, let us obtain $\phi_1(x)$ using Eq. (2.6). It is easy to see that

$$\begin{aligned}\phi_1(x) &= -i\hbar\frac{\delta \ln Z[J_1, J_2]}{\delta J_1(x)} \\ &= \frac{\langle 0(-\infty)|U_{J_2}(-\infty,+\infty)U_{J_1}(+\infty,x^0)\Phi_1(x)U_{J_1}(x^0,-\infty)|0(-\infty)\rangle}{Z[J_1,J_2]}, \end{aligned} \quad (2.11)$$

where x^0 is the time variable of $\phi_1(x)$ ($\phi_2(x)$ can be obtained similarly). Setting $J_1 = J_2 = J$ and using $Z[J,J] = 1$, we find that $\phi_1(x) = \phi_2(x) = \phi(x)$, where

$$\phi(x) = \langle 0(-\infty)|U_J(-\infty,x^0)\Phi(x)U_J(x^0,-\infty)|0(-\infty)\rangle. \quad (2.12)$$

If we interpret $U_J(x^0,-\infty)|0(-\infty)\rangle$, which is the state evolving from the vacuum state in the infinite past $|0(-\infty)\rangle$, as the vacuum state at time x^0, then $\phi(x)$ is simply the vacuum expectation value (mean field) of $\Phi(x)$ at time x^0. The pictorial interpretation of the above process of obtaining the mean field $\phi(x)$ is shown in Fig. 2.1, in which the condition $J_1 = J_2$ results in the disappearance of the contributions from things at time $t > x^0$ [2, 10]. This means that causality is automatic in this formalism.

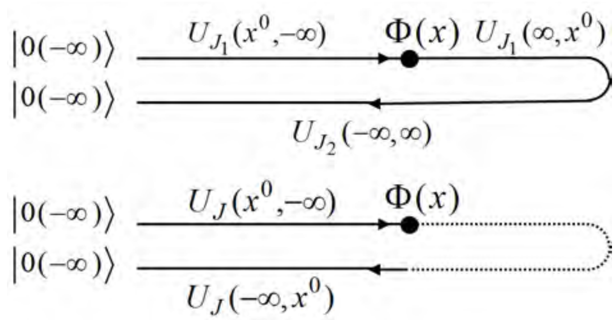

Figure 2.1: The pictorial interpretation of the mean-field evaluation. The first diagram represents Eq. (2.11), while the second one shows the cancelation of things at time $t > x^0$ after we set $J_1 = J_2$.

The path integral representation of Eq. (2.6) can be formulated using the following trick. We first enlarge the size of the time dimension by a factor of two, where x^0 runs from $-\infty$ to $+\infty$ in the first half, and from $+\infty$ back to $-\infty$ in the second half. (Thus the time dimension has the structure of a closed loop, hence the name "closed-time path integral.") As mentioned earlier, we denote the scalar field by Φ_1 (with the source J_1) in the first half of the time dimension, and by Φ_2 (with the source J_2) in the second half, so that the path integral representation of Eq. (2.6) takes the form[2]

$$Z[J_1, J_2,] = \int D\Phi_1 D\Phi_2 \exp\left\{\frac{i}{\hbar}[(S[\Phi_1] + J_1\Phi_1) - (S[\Phi_2] + J_2\Phi_2)]\right\} \quad (2.13)$$

subject to the conditions that $\Phi_1(+\infty) = \Phi_2(+\infty)$ and $J_1(+\infty) = J_2(+\infty)$. Note that the minus sign in front of the Φ_2 part of the exponent came from the fact that Φ_2 propagates backward in time while the integral over time is defined in the forward time direction.

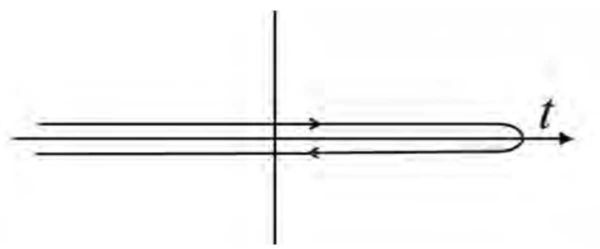

Figure 2.2: The closed-time contour used in the closed-time path integral.

Even though we formally have only one scalar field propagating along two time branches (one from $x^0 = -\infty$ to $+\infty$ and the other from $t = +\infty$ to $-\infty$), $Z[J_1, J_2]$ has the mathematical structure of a functional integral over two scalar fields, so we can evaluate it in the conventional way. Introducing the metric $c_{ab} = c^{ab} = diag(1, -1)$ $(a, b = 1, 2)$, we can use it to raise and lower the

[2]Here, we assume that Φ_a are real scalar fields. In the general situation in which the fields may be complex scalar fields or spinor fields, we have to write down the action for the backward time branch as $\int_{\infty}^{-\infty} dt \int d^3x \mathscr{L}(\Phi, \partial_\mu \Phi)$ first, and then change the time variable from t to $-t$ (this causes $\Phi(\vec{x}, t)$ in the integral to change to $\Phi(\vec{x}, -t)$). Expressing $\Phi(\vec{x}, -t)$ in terms of $\Phi(\vec{x}, t)$ via the time-reversal transformation results in the dependence on the complex conjugate of $\Phi(\vec{x}, t)$ of the Lagrangian density $\mathscr{L}$. Thus, we should write $S[\Phi_2^*]$ instead of $S[\Phi_2]$.

indices according to the rule $J^a = c^{ab} J_b$ and $\Phi_a = c_{ab}\Phi^b$. Using this metric, the generating functional can be written in the more compact form as

$$Z[J^a] = \int D\Phi_a \exp\left\{\frac{i}{\hbar}(S[\Phi_a] + J^a\Phi_a)\right\}, \tag{2.14}$$

where $S[\Phi_a] \equiv S[\Phi_1] - S[\Phi_2]$. The corresponding effective action is defined as a Legendre transform of $W[J^a] = -i\hbar \ln Z[J^a]$,

$$\Gamma[\phi_a] = W[J^a] - \phi_a J^a, \tag{2.15}$$

where $\phi_a(x) \equiv \delta W/\delta J^a(x)$. The rest of the calculation is the same as that in the conventional quantum field theory. As discussed earlier, the vacuum expectation value of the original scalar field is $\phi(x) = \delta W/\delta J^1(x)|_{J_1=J_2}$, which is obtained by solving the equation

$$\left.\frac{\delta\Gamma}{\delta\phi_a(x)}\right|_{\phi_1=\phi_2=\phi} = 0 \tag{2.16}$$

describing the time-evolution of $\phi(x)$ (remember that, unlike the conventional quantum field theory, $\phi(x)$ is time dependent in the nonequilibrium situation). It is worth mentioning that this time evolution should respect causality, since $\phi(x)$ depends on things that happened only in its past, by construction.

Let us now consider the propagators of the theory. We first recall that a propagator is the vacuum expectation value of the time-ordering product of two scalar fields. In the closed-time path integral formalism, this time ordering is defined to be along the direction of the closed-time path. As there are two scalar fields, one for each time branch, it is clear that there are four types of propagators [2]; the first two being formed by the fields on the same time branch, while the others being constructed from the fields on different time branches. In the functional method, these propagators are obtained by performing the functional differentiation on $W[J^a]$ [9],

$$iG_{ab}(x, x') = \frac{\delta^2 W[J^c]}{\delta J^b(x')\delta J^a(x)}. \tag{2.17}$$

Using the same algebra as we have done to find the mean field $\phi(x)$, we find

$$
\begin{aligned}
iG_{11}(x,x') &= \left.\frac{\delta^2 W[J^a]}{\delta J^1(x')\delta J^1(x)}\right|_{J_1=J_2=J} \\
&= \frac{i}{\hbar}\Big\{\langle 0(-\infty)|U_J(-\infty,x'^0)\Phi(x')U_J(x'^0,x^0)\Phi(x)U_J(x^0,-\infty)|0(-\infty)\rangle \\
&\quad - \langle 0(-\infty)|U_J(-\infty,x'^0)\Phi(x')U_J(x'^0,-\infty)|0(-\infty)\rangle \\
&\quad \times \langle 0(-\infty)|U_J(-\infty,x^0)\Phi(x)U_J(x^0,-\infty)|0(-\infty)\rangle\Big\} \\
&= \frac{i}{\hbar}\langle 0(-\infty)|T(\Phi(x')\Phi(x))|0(-\infty)\rangle_{connected}.
\end{aligned} \tag{2.18}
$$

Following the same procedure, the remaining propagators are found to be

$$
\begin{aligned}
iG_{12}(x,x') &= \left.\frac{\delta^2 W[J^a]}{\delta J^2(x')\delta J^1(x)}\right|_{J_1=J_2=J} \\
&= \frac{i}{\hbar}\langle 0(-\infty)|\Phi(x')\Phi(x))|0(-\infty)\rangle_{connected}\,,
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
iG_{21}(x,x') &= \left.\frac{\delta^2 W[J^a]}{\delta J^2(x)\delta J^1(x')}\right|_{J_1=J_2=J} \\
&= \frac{i}{\hbar}\langle 0(-\infty)|\Phi(x)\Phi(x'))|0(-\infty)\rangle_{connected}\,,
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
iG_{22}(x,x') &= \left.\frac{\delta^2 W[J^a]}{\delta J^2(x')\delta J^2(x)}\right|_{J_1=J_2=J} \\
&= \frac{i}{\hbar}\langle 0(-\infty)|\tilde{T}(\Phi(x)\Phi(x'))|0(-\infty)\rangle_{connected}.
\end{aligned} \tag{2.21}
$$

Observe that $G_{22}(x,x')$ is defined using the reverse time ordering, due to the fact that it contains only the fields on the reverse time branch (on which time runs backward), while the time-ordering operator does not appear in G_{12} and G_{21} since these propagators contains fields on different time branches (the field on the first branch must be on the right for these propagators). Thus, unlike the conventional quantum field theory in which only the Feynman propagator exists, there are four propagators that contribute to each internal line of a Feynman diagram. Since each interaction vertex is defined on either forward or reverse time branch, it is not hard to see that a vacuum Feynman diagram with n vertices in the conventional quantum field theory may be thought of as describing 2^n diagrams in the CTP formalism.

The above set of propagators was obtained naively from the formalism, but their physical meaning may not be clear. To make the physical sense out of these propagators, it is appropriate to define another set of propagators. To do so, we first observe that that G_{ab} can be written in terms of the step function $\Theta(t-t')$ which is equal to 1 when $t>t'$ and zero otherwise. For example,

$$\hbar G_{11} = \langle 0(-\infty)|\left(\Phi(x)\Phi(x')\Theta(x^0-x'^0)+\Phi(x')\Phi(x)\Theta(x'^0-x^0)\right)|0(-\infty)\rangle \quad (2.22)$$

$$\hbar G_{22} = \langle 0(-\infty)|\left(\Phi(x')\Phi(x)\Theta(x^0-x'^0)+\Phi(x)\Phi(x')\Theta(x'^0-x^0)\right)|0(-\infty)\rangle. \quad (2.23)$$

Using the identity $\Theta(x^0-x'^0)+\Theta(x'^0-x^0)=1$ and noting that G_{ab}'s satisfy the constraint $G_{11}+G_{22}-G_{12}-G_{21}=0$ (which implies that there are only three independent propagators), we define the retarded propagator (G_R), the advanced propagator (G_A) and the correlation propagator (G_C) by [10]

$$\begin{aligned} G_R(x,x') &= G_{11}(x,x')-G_{12}(x,x') \\ &= \frac{1}{\hbar}\Theta(x^0-x'^0)\langle 0(-\infty)|[\Phi(x),\Phi(x')]|0(-\infty)\rangle\,, \quad (2.24) \\ G_A(x,x') &= G_{11}(x,x')-G_{21}(x,x') \\ &= -\frac{1}{\hbar}\Theta(x'^0-x^0)\langle 0(-\infty)|[\Phi(x),\Phi(x')]|0(-\infty)\rangle\,, \quad (2.25) \\ G_C(x,x') &= G_{12}+G_{21} \\ &= \frac{1}{\hbar}\langle 0(-\infty)|\{\Phi(x),\Phi(x')\}|0(-\infty)\rangle. \quad (2.26) \end{aligned}$$

In practice, it is appropriate to express all equations of motion in terms of these propagators, so as to make causality apparent.

Having mentioned that the effective action $\Gamma[\phi]$ is an important quantity for obtaining the mean fields, let us now discuss how we can obtain it in practice. The standard method for this purpose is known as the "background field" technique, in which one expresses the quantum field $\Phi(x)$ as a sum of a mean field $\phi(x)$ and a fluctuation field $\varphi(x)$, $\Phi(x)=\phi(x)+\varphi(x)$. Here, $\phi(x)$ is treated as a function whose explicit form will be determined from the equations of motion, and

$\varphi(x)$ is a quantum field whose (quantum-corrected) vacuum expectation value is tuned to zero by adding the appropriate counter terms when doing quantum calculations. To apply this method in the closed-time path integral formalism, we first use Eqs. (2.14) and (2.15) to express the effective action in the form of a functional integral:

$$\Gamma[\phi_a] = -i\hbar \ln \left[\int D\Phi_a \exp \left(\frac{i}{\hbar} \left\{ S[\Phi_a] - \frac{\delta \Gamma}{\delta \phi_a} (\Phi_a - \phi_a) \right) \right\} \right] , \tag{2.27}$$

where we have expressed the sources as $J^a = -\delta\Gamma/\delta\phi_a$. Decomposing $\Phi_a = \phi_a + \varphi_a$, the functional integral over Φ_a becomes the one over φ_a. By imposing the initial conditions that $\phi_a(t = -\infty) = \Phi_a(-\infty)$ (which will be imposed when solving the equations $\delta\Gamma/\delta\phi_a(x) = 0$ for determining $\phi_a(x)$), we see that φ_a vanishes at $t = -\infty$. If we also assume that φ_a also vanishes at $t = +\infty$, then we can evaluate the functional integral over φ_a in the same way as we did in the conventional quantum field theory without having to worry about the constraints due to the initial conditions. With this field decomposition, we have

$$\begin{aligned} \Gamma[\phi_a] &= -i\hbar \ln \left[\int D\varphi_a \exp \left\{ \frac{i}{\hbar} \left(S[\phi_a] + \left(\frac{\delta S}{\delta \phi_a} - \frac{\delta \Gamma}{\delta \phi_a} \right) \varphi_a \right.\right.\right. \\ &\quad \left.\left.\left. + \frac{1}{2} \frac{\delta^2 S}{\delta \phi_a \delta \phi_b} \varphi_a \varphi_b + S_Q \right) \right\} \right] \\ &\equiv S[\phi_a] + \Gamma_1[\phi_a], \end{aligned} \tag{2.28}$$

where we have expanded the action $S[\phi_a + \varphi_a]$ in terms of the fluctuation fields

$$S[\phi_a + \varphi_a] = S[\phi_a] + \frac{\delta S}{\delta \phi_a} \varphi_a + \frac{1}{2} \frac{\delta^2 S}{\delta \phi_a \delta \phi_b} \varphi_a \varphi_b + S_Q \, , \tag{2.29}$$

with S_Q being the part of the action containing the higher-order terms in the fluctuation fields, and $\Gamma_1[\phi_a]$ defined by

$$\Gamma_1[\phi_a] = -i\hbar \ln \left[\int D\varphi_a \exp \left\{ \frac{i}{\hbar} \left(\frac{1}{2} \frac{\delta^2 S}{\delta \phi_a \delta \phi_b} \varphi_a \varphi_b + S_Q - \frac{\delta \Gamma_1}{\delta \phi_a} \varphi_a \right) \right\} \right] \tag{2.30}$$

represents the quantum corrections to the classical action $S[\phi_a]$ (that $\Gamma[\phi_a]$ is of this form is the main reason why it was named the effective action). In obtaining the above form of Γ_1, we have used $\delta\Gamma/\delta\phi_a = \delta S/\delta\phi_a + \delta\Gamma_1/\delta\phi_a$ to express the right-hand side in terms of the derivatives of Γ_1. By comparing Eq. (2.30) with Eq. (2.28), we can interpret Γ_1 as the effective action of the theory of φ_a coupled to the external sources $\delta\Gamma_1/\delta\phi_a$, whose action is $S_r \equiv (1/2)(\delta^2 S/\delta\phi_a\delta\phi_b)\varphi_a\varphi_b + S_Q$.

Let us now show that the vacuum expectation value of φ_a vanishes. Performing the functional derivative of Eq. (2.30) with respect to ϕ_c, we get

$$\begin{aligned} 0 &= \int D\varphi_a \frac{i}{\hbar}\left(\frac{1}{2}\frac{\delta^3 S}{\delta\phi_c\delta\phi_b\delta\phi_a}\varphi_a\varphi_b + \frac{\delta S_Q}{\delta\phi_c} - \frac{\delta\Gamma_1}{\delta\phi_c}\right. \\ &\qquad \left. - \frac{\delta^2\Gamma_1}{\delta\phi_c\delta\phi_a}\varphi_a\right)\exp\left\{\frac{i}{\hbar}\left(\frac{1}{2}\frac{\delta^2 S}{\delta\phi_a\delta\phi_b}\varphi_a\varphi_b + S_Q - \frac{\delta\Gamma_1}{\delta\phi_a}\varphi_a\right)\right\}. \end{aligned} \tag{2.31}$$

Using $S_Q = 1/6\{\delta^3 S/(\delta\phi_a\delta\phi_b\delta\phi_c)\}\varphi_a\varphi_b\varphi_c + \ldots$, one can verify that

$$\frac{1}{2}\frac{\delta^3 S}{\delta\phi_c\delta\phi_b\delta\phi_a}\varphi_a\varphi_b + \frac{\delta S_Q}{\delta\phi_c} = \frac{\delta}{\delta\varphi_c}\left(\frac{1}{2}\frac{\delta^2 S}{\delta\phi_a\delta\phi_b}\varphi_a\varphi_b + S_Q\right) - \frac{\delta^2 S}{\delta\phi_c\delta\phi_b}\varphi_b. \tag{2.32}$$

Substituting this result into Eq. (2.31) and using $S + \Gamma_1 = \Gamma$, we obtain

$$0 = -\frac{\delta^2\Gamma}{\delta\phi_c\delta\phi_a}\langle\varphi_a\rangle + \int D\varphi_a\frac{\delta}{\delta\varphi_c}\left(\exp\left\{\frac{i}{\hbar}\left(\frac{1}{2}\frac{\delta^2 S}{\delta\phi_a\delta\phi_b}\varphi_a\varphi_b + S_Q - \frac{\delta\Gamma_1}{\delta\phi_a}\varphi_a\right)\right\}\right) \tag{2.33}$$

where $\langle\varphi_a\rangle$ is the vacuum expectation value of φ_a. Since the integral on the right-hand side is zero and the matrix $\delta^2\Gamma/\delta\phi_a\delta\phi_b$ cannot be singular (because of the identity $(\delta^2\Gamma/\delta\phi_a\delta\phi_b)(\delta^2 W/\delta J^b\delta J^c) = \delta_c{}^a$) [10], then we must have $\langle\varphi_a\rangle = 0$ and so the vacuum expectation values of the fluctuation fields vanish.

It is important to make a remark that even though there are two dynamical equations for the mean fields $\delta\Gamma/\delta\phi_a = 0$, these equations are supposed to yield only one dynamical equation for the mean field in the coincidence limit, in which $\phi_1 = \phi_2 \equiv \phi$ and $J_1 = J_2$, otherwise we would get a nonsense result. This requirement puts a constraint on the form of the effective action, which we now explore. Recall that we have an identity $Z[J, J] = 1$ (see Eq. (2.6)), which implies

$$\Gamma[\phi,\phi] = -i\hbar\,\ln Z[J,J] - J\,\phi + J\,\phi = 0. \tag{2.34}$$

The effective action thus vanishes in the coincidence limit and this puts a constraint on the form of $\Gamma[\phi_a]$ as follows. Introducing the new field variables $\triangle(x)$ and $\Sigma(x)$ defined by [1, 10]

$$\triangle(x) = \phi_1(x) - \phi_2(x) \tag{2.35}$$

$$\Sigma(x) = \frac{1}{2}(\phi_1(x) + \phi_2(x)). \tag{2.36}$$

In the coincidence limit, we have $\triangle(x) = 0$ and $\Sigma(x) = \phi(x)$. In terms of these fields, the general form of the effective action is

$$\begin{aligned}\Gamma[\triangle, \Sigma] &= \triangle(x)N_1(x) + \Sigma(x)M_1(x) + \frac{1}{2}\triangle(x)N_2(x,x')\triangle(x') \\ &+\frac{1}{2}\Sigma(x)M_2(x,x')\Sigma(x') + \frac{1}{2}\Sigma(x)D_2(x,x')\triangle(x') + \ldots .\end{aligned} \tag{2.37}$$

The requirement that Γ mush vanish in the coincidence limit rules out the possibility of having terms containing only Σ in the effective action, which means that M_1 and M_2 in the above equation have to vanish. This result implies that the original two dynamical equations give only one independent dynamical equation in the coincidence limit as follows. In terms of Σ and Δ, the dynamical equations are $\delta\Gamma/\delta\Sigma = 0$ and $\delta\Gamma/\delta\Delta = 0$. It is easy to see that, with the constraint on the form of Γ just stated, the equation $\delta\Gamma/\delta\Sigma = 0$ is trivial in the coincidence limit, and so the only nontrivial dynamical equation for determining the mean field is $(\delta\Gamma/\delta\Delta)|_{\Sigma=\phi,\Delta=0} = 0$.

We end this section with the discussion about how one can incorporate the initial conditions into the closed-time path integral. When dealing with the nonequilibrium problem, we are normally given an information about the probability distribution of the initial states (at $t = -\infty$), and then asked what will happen in the future given the initial conditions. It is clear that this initial probability distribution is the thing that was missing in the above formulation of the closed-time path integral, which is supposed to describe the nonequilibrium system. This remark tells us that the CTP generating functional introduced previously has to be modified in order to completely describe the nonequilibrium system. It turns out that the correct CTP generating functional takes the form [3, 6]

$$Z[J_1, J_2, \rho] = \mathrm{Tr}[U_{J_2}(-\infty, +\infty)U_{J_1}(+\infty, -\infty)\rho(-\infty)], \tag{2.38}$$

where $\rho(-\infty)$ is the density operator at the initial time, which contains all information about the probability distribution of the initial states [6], and the trace is taken over all initial states. The corresponding path integral representation is found to be

$$\begin{aligned} Z[J^a, \rho] &= \int D\Phi_1(-\infty)D\Phi_2(-\infty)\left[\langle\Phi_1(-\infty)|\rho|\Phi_2(-\infty)\rangle\right. \\ &\qquad \left.\times \int D\Phi_a \exp\left\{\frac{i}{\hbar}[(S[\Phi_a] + J^a\Phi_a)]\right\}\right], \end{aligned} \tag{2.39}$$

where $\langle\Phi_1(-\infty)|\rho|\Phi_2(-\infty)\rangle$ is the matrix element of the density operator with respect to the initial states, the functional integral over Φ_a is subject to the constraint that each field approaches its given initial state $\Phi_a(-\infty)$ as $t \to -\infty$, and the functional integration over the initial states ($\Phi_1(-\infty)$ and $\Phi_2(-\infty)$) is performed after the integral over Φ_a has been evaluated. Evaluating the above form of the functional integral is surely a formidable task. Luckily enough, we will not have to deal with such a messy functional integral in this thesis, and will assume that the system was initially in a pure state from now on.

2.2 2-Particle Irreducible Formalism

In this section, we will introduce the 2-particle irreducible formalism, which plays an important role in the nonequilibrium quantum field theory. To begin with, let us add an exponential factor [3, 5, 6]

$$\exp\left(\frac{i}{\hbar}K[\Phi_a]\right) \tag{2.40}$$

into the path integral, where

$$K[\Phi_a] = K + K^a(x)\Phi_a(x) + \frac{1}{2}K^{ab}(x,x')\Phi_a(x)\Phi_b(x') + \ldots . \tag{2.41}$$

In the above expression for $K[\Phi_a]$, the integration over all spacetime coordinates should be understood without having to write it, and the kernels K's with more than one index ($K^{ab}(x,x'), K^{abc}(x,x',x''), \ldots$) are nonlocal in the sense that they couple the fields at different spacetime points. With this exponential factor, we define a new generating functional $Z[J^a, K^{ab}, \ldots]$ by

$$\begin{aligned} Z[J^a, K^{ab}, ..] &= \int D\Phi_a \exp\Big[\frac{i}{\hbar}\Big\{S[\Phi_a] + J^a(x)\Phi_a(x) + \frac{1}{2}K^{ab}(x,x')\Phi_a(x)\Phi_b(x') \\ &\quad + \frac{1}{6}K^{abc}(x,x',x'')\Phi_a(x)\Phi_b(x')\Phi_c(x'') + \ldots\Big\}\Big], \end{aligned} \tag{2.42}$$

where we have absorbed K into the normalization factor and K^a into the external sources J^a. The connected generating functional is defined in the usual way as $W[J^a, K^{ab}, \ldots] \equiv -i\hbar \ln Z[J^a, K^{ab}, \ldots]$ but with non-local sources K's, and the effective action is defined by performing the multiple Legendre transformation on $W[J^a, K^{ab}, \ldots]$ as [3, 6, 10]

$$\begin{aligned} \Gamma[\phi_a, G_{ab}, \ldots] &= W[J^a, K^{ab}, \ldots] - J^a\phi_a - \frac{1}{2}K^{ab}(\phi_a\phi_b + \hbar G_{ab}) \\ &\quad -\frac{1}{6}K^{abc}(\phi_a\phi_b\phi_c + \phi_a\hbar G_{bc} + \phi_b\hbar G_{ac} + \phi_c\hbar G_{ab} + \hbar^{\frac{3}{2}}G_{abc}) \\ &\quad - \ldots, \end{aligned} \tag{2.43}$$

where the mean field $\phi_a(x)$ is defined by

$$\frac{\delta W}{\delta J^a(x)} = \phi_a(x) \tag{2.44}$$

and the nonlocal kernels ($G_{ab}(x,x'), G_{abc}(x,x',x''), \ldots$) are defined by

$$\frac{\delta W}{\delta K^{ab}(x,x')} = \frac{1}{2}\Big\{\phi_a(x)\phi_b(x') + \hbar G_{ab}(x,x')\Big\} \tag{2.45}$$

$$\begin{aligned} \frac{\delta W}{\delta K^{abc}(x,x',x'')} &= \frac{1}{6}\Big\{\phi_a(x)\phi_b(x')\phi_c(x'') + \phi_a(x)\hbar G_{bc}(x',x'') + \phi_b(x')\hbar G_{ac}(x,x'') \\ &\quad + \phi_c(x'')\hbar G_{ab}(x,x') + \hbar^{\frac{3}{2}}G_{abc}(x,x',x'')\Big\}, \end{aligned} \tag{2.46}$$

and so on. It is not hard to see that, with the form of the generating functional in Eq. (2.42), $\phi_a(x)$ is still the vacuum expectation value of $\Phi_a(x)$, while the nonlocal kernels G's are the connected n-point functions ($n \geq 2$) once all the external sources have been set to zero. The corresponding inverse transformations read

$$\begin{aligned} \frac{\delta\Gamma}{\delta\phi_a(x)} &= -J^a(x) - \phi_b(x')K^{ab}(x,x') \\ &\quad -\frac{1}{2}K^{abc}(x,x',x'')\{\hbar G_{bc}(x',x'') + \phi_b(x')\phi_c(x'')\} \\ &\quad -\ldots, \end{aligned} \tag{2.47}$$

$$\frac{\delta\Gamma}{\delta G_{ab}(x,x')} = -\frac{1}{2}\hbar K^{ab}(x,x') - \frac{1}{2}\hbar K^{abc}(x,x',x'')\phi_c(x'') - \ldots, \tag{2.48}$$

$$\frac{\delta\Gamma}{\delta G_{abc}(x,x',x'')} = -\frac{1}{6}\hbar^{\frac{3}{2}}K^{abc}(x,x',x'') - \ldots, \tag{2.49}$$

and so on. By setting all the external sources to zero in the above equations, we obtain a set of (infinite) equations which determines the mean fields and all the n-point functions (or correlation functions) [3]. As all information about a quantum field theory is contained in the n-point functions, it would be wonderful if we could solve the above set of equations and obtain all the n-point functions of the theory. But since the number of these equations is infinite, this method does not work in practice. An obvious way to improve the situation is to truncate the infinite series in Eq. (2.41) at the nth power of Φ_a,

$$K[\Phi_a] = K + K^a(x)\Phi_a(x) + \ldots + \frac{1}{n!}K^{a_1\cdots a_n}(x_1,\cdots,x_n)\Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n), \tag{2.50}$$

which results in a finite set of equations of motion for determining $\phi_a, G_{a_1a_2}, \ldots,$ $G_{a_1\cdots a_n}$, leaving the higher-order correlation functions undetermined. Such a formalism is known as the "n-particle-irreducible formalism."

Let us consider the simplest case of the "2-particle-irreducible formalism," in which there are two external sources J^a and K^{ab}, in detail. Applying this formalism to the closed-time path integral, the CTP generating functional with external sources J^a and K^{ab} takes the form

$$Z[J^a, K^{ab}] = \int D\Phi_a \exp\left[\frac{i}{\hbar}\left\{S[\Phi_a] + J^a(x)\Phi_a(x) + \frac{1}{2}K^{ab}(x,x')\Phi_a(x)\Phi_b(x')\right\}\right]. \tag{2.51}$$

The corresponding effective action is

$$\Gamma[\phi_a, G_{ab}] = W[J^a, K^{ab}] - J^a\phi_a - \frac{1}{2}K^{ab}(\phi_a\phi_b + G_{ab}), \tag{2.52}$$

where $W[J^a, K^{ab}] = i\hbar \ln Z[J^a, K^{ab}]$, and the equations of motion are

$$\frac{\delta\Gamma}{\delta\phi_a(x)} = -J^a(x) - \phi_b(x')K^{ab}(x,x'), \tag{2.53}$$

$$\frac{\delta\Gamma}{\delta G_{ab}(x,x')} = -\frac{1}{2}\hbar K^{ab}(x,x'). \tag{2.54}$$

Let us now use the background field technique to evaluate the above effective action. We start with a functional-integral form of the effective action,

$$\begin{aligned}\exp\left(\frac{i}{\hbar}\Gamma[\phi_a, G_{ab}]\right) &= \int D\Phi_a \exp\Big(\frac{i}{\hbar}\Big\{S[\Phi_a] + J^a(\Phi_a - \phi_a) \\ &\quad + \frac{1}{2}K^{ab}(\Phi_a\Phi_b - \phi_a\phi_b - \hbar G_{ab})\Big\}\Big).\end{aligned} \tag{2.55}$$

By decomposing the field operators as $\Phi_a(x) = \phi_a(x) + \varphi_a(x)$ and performing the same kind of calculation like what we did in the previous section, we obtain the result

$$\begin{aligned}\exp\left(\frac{i}{\hbar}\Gamma[\phi_a, G_{ab}]\right) &= \int D\varphi_a \exp\Big(\frac{i}{\hbar}\Big\{ S[\phi_a + \varphi_a] + J^a\varphi_a \\ &\quad + \frac{1}{2}K^{ab}(\varphi_a\varphi_b - 2\phi_a\varphi_b - \hbar G_{ab})\Big\}\Big). \\ &= \int D\varphi_a \exp\Big(\frac{i}{\hbar}\Big\{ S[\phi_a] + \left[\frac{\delta S}{\delta\phi_a} - \frac{\delta\Gamma}{\delta\phi_a}\right]\varphi_a \\ &\quad + \frac{1}{2}\left[\frac{\delta^2 S}{\delta\phi_a\delta\phi_b} - \frac{2}{\hbar}\frac{\delta\Gamma}{\delta G_{ab}}\right]\varphi_a\varphi_b \\ &\quad + \frac{\delta\Gamma}{\delta G_{ab}}G_{ab} + S_Q\Big\}\Big),\end{aligned} \tag{2.56}$$

where we have expanded the action as in Eq. (2.29). The above result can be rewritten as

$$\Gamma[\phi_a, G_{ab}] = S[\phi_a] + \frac{\delta\Gamma}{\delta G_{ab}}G_{ab} + \tilde{\Gamma}_2[\phi_a, G_{ab}] \tag{2.57}$$

where $\tilde{\Gamma}_2$ takes the form

$$\tilde{\Gamma}_2[\phi_a, G_{ab}] = -i\hbar \ln\left(\int D\varphi \exp\frac{i}{\hbar}\left(\left(\frac{\delta S}{\delta\phi_a} - \frac{\delta\Gamma}{\delta\phi_a}\right)\varphi_a + \frac{1}{2}\left(\frac{\delta^2 S}{\delta\phi_a\delta\phi_b} - \frac{2}{\hbar}\frac{\delta\Gamma}{\delta G_{ab}}\right)\varphi_a\varphi_b + S_Q\right)\right). \tag{2.58}$$

Just like $\Gamma_1[\phi_a]$ introduced in the previous section, $\Gamma_2[\phi_a, G_{ab}]$ is the effective action of a new theory in which the field variables are the fluctuation fields φ_a. However, unlike Γ_1, the propagator of this theory does not correspond to the whole of $(\delta^2 S/\delta\phi_a\delta\phi_b - (2/\hbar)\delta\Gamma/\delta G_{ab})$. The point is that, in the presence of $\delta\Gamma/\delta G_{ab}$, it is expected that the action should contain some nonlocal bilinear terms (that is, the terms of the form $\int d^4x \int d^4x'\, \tilde{K}^{ab}(x, x')\phi_a(x)\phi_b(x')$)[3] and these terms cannot be included as a part of the propagator. This motivates us to write

$$\frac{\delta^2 S}{\delta\phi_a\delta\phi_b} - \frac{2}{\hbar}\frac{\delta\Gamma}{\delta G_{ab}} = iG^{-1}_{ab} + \tilde{K}^{ab}, \tag{2.59}$$

where the inverse propagator G^{-1}_{ab} and $\tilde{K}^{ab}$ are the local and the nonlocal parts, respectively. Using the above decomposition, Eq. (2.57) becomes

$$\Gamma[\phi_a, G_{ab}] = S[\phi_a] + \frac{\hbar}{2}\frac{\delta^2 S}{\delta\phi_a\delta\phi_b}G_{ab} - \frac{\hbar}{2}\tilde{K}^{ab}G_{ab} + \tilde{\Gamma}_2[\phi_a, G_{ab}] + const. \tag{2.60}$$

Before we continue, let us recall one important fact about the effective action $\Gamma[\phi_a]$ of the previous section: $\Gamma[\phi_a]$ can be expressed as a sum of all vacuum loop diagrams, where the diagrams containing n loops are multiplied by $\hbar^n$. Such a property is expected to hold for $\Gamma[\phi_a, G_{ab}]$ under consideration. As the second term on the right-hand side of Eq. (2.60) is multiplied by the first power of $\hbar$, it must represent the sum of 1-loop diagrams.[4] The rest of 1-loop diagrams is contained in $\tilde{\Gamma}_2$. As it is well known that only the term $G^{-1}_{ab}\phi_a\phi_b$ of the action

[3]Since there are non-local source terms in our formalism, we expect that the theory of fluctuation fields should possess the same structure.

[4]The third term of the form $\tilde{K}^{ab}G_{ab}$ contains a nonlocal kernel $\tilde{K}^{ab}$, and so cannot describe any loop diagram.

contributes to the 1-loop diagrams, we conclude that the 1-loop part of $\tilde{\Gamma}_2$ takes the form

$$-i\hbar \ln\Big\{ \int D\varphi \exp\left[\frac{i}{\hbar}\left(\frac{1}{2}(iG^{-1}_{ab}\varphi_a\varphi_b)\right)\right]\Big\} = -\frac{i\hbar}{2}\ln\,\det(\hbar G), \tag{2.61}$$

which is just $\tilde{\Gamma}_2$ without interactions and external sources. Thus, after removing the 1-loop part from $\tilde{\Gamma}_2$, we can write

$$\Gamma[\phi_a, G_{ab}] = S[\phi_a] + \frac{\hbar}{2}\frac{\delta^2 S}{\delta\phi_a\delta\phi_b}G_{ab} - \frac{i\hbar}{2}\ln\,\det G + \Gamma_2[\phi_a, G_{ab}] + const., \tag{2.62}$$

where

$$\begin{aligned}\Gamma_2[\phi_a, G_{ab}] \;\equiv\; & -i\hbar\ln\left\{\frac{1}{\det(\hbar G)^{\frac{1}{2}}}\int D\varphi_a \exp\left(-\frac{1}{2}(\hbar G_{ab})^{-1}\varphi_a\varphi_b\right.\right. \\ & \left.\left. +\frac{i}{\hbar}\left(S_Q + \tilde{J}^a\varphi_a + \frac{1}{2}\tilde{K}^{ab}(\varphi_a\varphi_b - \hbar G_{ab})\right)\right)\right\} \end{aligned} \tag{2.63}$$

represents the diagrams with more than one loop. In the above equation, the source terms are

$$\tilde{J}^a \;=\; -\left(\frac{\hbar}{2}\frac{\delta^3 S}{\delta\phi_a\delta\phi_b\delta\phi_c}G_{bc} + \frac{\delta\Gamma_2}{\delta\phi_a}\right) \tag{2.64}$$

$$\tilde{K}^{ab} \;=\; -\frac{2}{\hbar}\frac{\delta\Gamma_2}{\delta G_{ab}}, \tag{2.65}$$

where we have used Eq. (2.62) to express the derivatives of $\Gamma[\phi_a, G_{ab}]$ in terms of the derivatives of $\Gamma_2[\phi_a, G_{ab}]$ and some other quantities.

Let us now describe how one can calculate Γ_2 in practice. Taking the derivative of Eq. (2.62) with respect to G_{ab}, we obtain

$$iG^{-1}_{ab} \;=\; \frac{\delta^2 S}{\delta\phi_a\delta\phi_b} + K^{ab} + \frac{2}{\hbar}\frac{\delta\Gamma_2}{\delta G_{ab}}. \tag{2.66}$$

Setting $K^{ab} = 0$, we see that the inverse propagator iG^{-1}_{ab} is the sum of a free inverse propagator $\delta^2 S/\delta\phi_a\delta\phi_b$ and $(2/\hbar)\delta\Gamma_2/\delta G_{ab}$, and so we conclude that $(2/\hbar)\delta\Gamma_2/\delta G_{ab}$ corresponds to the self-energy terms. Since the vacuum expectation values of the fluctuation fields φ_a vanish, we conclude that iG^{-1}_{ab} is a sum of 1-particle-irreducible (1PI) diagrams. And since $\delta^2 S/\delta\phi_a\delta\phi_b$ is a "constant" term in the sense that it represents a Feynman diagram with a single line, Eq. (2.66) implies that $\delta\Gamma_2/\delta G_{ab}$ must also be the sum of 1PI diagrams. This leads us to conclude that Γ_2 is the sum of 2-particle-irreducible (2PI) diagrams (the diagrams that cannot be separated into two parts by cutting two internal lines) with respect to the propagators G_{ab} [5], so that $\Gamma_2[\phi_a, G_{ab}]$ is called the 2PI effective action. In practice, we calculate $\Gamma_2[\phi_a, G_{ab}]$ using the formula

$$\Gamma_2[\phi_a, G_{ab}] = -i\hbar \times \{\text{the sum of all 2PI vacuum Feynman diagrams}\}\,. \quad (2.67)$$

We thus can practically obtain Γ_2 by calculating 2PI diagrams order by order in perturbation; this motivated the name "2-particle-irreducible formalism."

Once $\Gamma_2[\phi_a, G_{ab}]$ has been evaluated, we can substitute the resulting effective action back into Eqs. (2.53) and (2.54), and obtain the equations for determining ϕ_a and G_{ab} by setting $J^a = K^{ab} = 0$. By setting $\phi_1 = \phi_2 \equiv \phi$, the equations for determining the mean fields and the propagators are obtained.

CHAPTER III

NONEQUILIBRIUM DYNAMICS OF THE INFLATON

In this chapter, we will apply the method of nonequilibrium quantum field theory to study the dynamics of the inflaton field which drives inflation in cosmology. We will begin with a brief review on inflation, and then go on to derive the dynamical equations of a scalar field theory coupled to gravity. Finally, we will consider a specific case of the inflaton in the Friedmann-Robertson-Walker (FRW) spacetime and derive the dynamical equations for the inflaton field and its propagator in the nonequilibrium setting.

3.1 A Brief Review on Inflation

The inflation is the stage of the early universe in which gravity acts as a repulsive force and causes the universe to expand with acceleration. By including the inflation, some important problems of the Hot Big Bang model such as the horizon problem and the flatness problem are solved [11, 12, 13]. Because of the success of the prediction of the Hot Big Bang model, inflation must begin and end in the very early stage of the universe. However, the duration of the inflation should be long enough to solve the problems of the Hot Big Bang model.

We are interested in the spacetime described by the flat FRW metric, which takes the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix} \tag{3.1}$$

where the scale factor $a(t)$ describes the spatial expansion of spacetime. The corresponding Ricci tensor $R_{\mu\nu}$, scalar curvature R and Einstein tensor $G_{\mu\nu}$ are

$$R_{00} = -3\frac{\ddot{a}}{a}\,, \tag{3.2}$$

$$R_{ij} = (2\dot{a}^2 + a\ddot{a})\delta_{ij}\,, \tag{3.3}$$

$$R = -6\Big(\frac{\ddot{a}}{a} + \Big(\frac{\dot{a}}{a}\Big)^2\Big)\,, \tag{3.4}$$

$$G_{00} = 3\Big(\frac{\dot{a}}{a}\Big)^2\,, \tag{3.5}$$

$$G_{ij} = -(\dot{a}^2 + 2a\ddot{a})\delta_{ij}\,, \tag{3.6}$$

where $\dot{a} \equiv da/dt$. The above form of the Einstein tensor implies that there must be only two independent equations of motion coming from the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. The energy-momentum tensor $T_{\mu\nu}$ that produces the FRW metric, therefore, must have two independent components and takes the perfect-fluid form

$$T_{00} = \rho\,, \tag{3.7}$$

$$T_{ij} = p\,a^2\,\delta_{ij}, \tag{3.8}$$

where ρ and p are the energy density and the pressure of the fluid, respectively. Using the FRW metric and the above energy-momentum tensor, we obtain the Friedmann equations from the Einstein equation, which describe the dynamics of the scale factor,

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho + 3\,p) \tag{3.9}$$

$$H^2 = \frac{8\pi G}{3}\rho, \tag{3.10}$$

where $H \equiv \dot{a}/a$ is called the Hubble parameter. From the Friedmann equations, we see that if ρ and p are positive, the spacetime will expand with deceleration. For the Friedmann model to describe the state of inflation, we have to allow the term $(\rho + 3p)$ to become negative during some period of time in order to create the accelerated expanding universe. For example, in the vacuum dominated era in which the only contribution to the energy-momentum tensor is from the cosmological constant Λ,

$$T^{\Lambda}_{\mu\nu} = \frac{\Lambda}{8\pi G}g_{\mu\nu}, \tag{3.11}$$

we have $\rho = \Lambda/(8\pi G)$ and $p = -\rho$, by using Eqs. (3.1), (3.7) and (3.8). The corresponding dynamical equation for the scale factor reads

$$\ddot{a} = \frac{8\pi}{3} G\rho a \equiv H^2 a, \tag{3.12}$$

whose solution is

$$a = e^{Ht}, \tag{3.13}$$

which describes the exponentially expanding universe. However, as time passes, the condition $(\rho+3p)$ must evolve into a positive value to recover the decelerated universe in agreement with the standard cosmological prediction [11, 12]. To allow inflation to occur, we need the matter that contributes the negative pressure to the energy-momentum tensor. The simplest one that does this job is of the form of a scalar field, known as the inflaton field. The action of the inflaton is normally of the form

$$S[\phi] = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right), \tag{3.14}$$

where ϕ is the inflaton field and $V(\phi)$ is the potential term of the inflaton. The corresponding equation of motion is obtained by varying the action with respect to the inflaton field,

$$\Box \phi + V'(\phi) = 0, \tag{3.15}$$

where $\Box$ is the d' Alembertian operator $\Box = g^{\mu\nu}\nabla_\mu \nabla_\nu$ and $V'(\phi) \equiv dV(\phi)/d\phi$. To find the explicit form of the d' Alembertian operator in the case of FRW spacetime, we first recall that $\Box = g^{\mu\nu}\nabla_\mu \nabla_\nu$, and so its operation on ϕ is

$$g^{\mu\nu}\nabla_\mu \nabla_\nu \phi = g^{\mu\nu}\{\partial_\mu \partial_\nu \phi - \Gamma^\rho_{\mu\nu} \partial_\rho \phi\}. \tag{3.16}$$

From the identity $g^{\mu\sigma} g_{\sigma\nu} = \delta_\nu{}^\mu$ and Eq. (3.1), we have $g^{\mu\nu} = diag\{1, -a^{-2}(t), -a^{-2}(t), -a^{-2}(t)\}$. Using the definition of the Christoffel symbol,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}), \tag{3.17}$$

we find that the non-zero components are

$$\Gamma^{i}{}_{j0} = \frac{\dot{a}}{a}\delta_j{}^i$$
$$\Gamma^{0}{}_{ij} = a\dot{a}\delta_{ij}, \qquad (3.18)$$

where $i, j = 1, 2, 3$ are the spatial indices. We thus obtain

$$\begin{aligned} g^{\mu\nu}\nabla_\mu\nabla_\nu\phi &= g^{\mu\nu}\partial_\mu\partial_\nu\phi - g^{ii}\Gamma^0_{jj}\delta^j{}_i\frac{d\phi}{dt} \\ &= \frac{d^2\phi}{dt^2} + 3\frac{\dot{a}(t)}{a(t)}\frac{d\,\phi}{dt}, \end{aligned} \qquad (3.19)$$

where we have assumed that the inflaton field is spatially independent since we are interested in the spatially homogeneous and isotropic universe (so that we can neglect the spatial derivative terms). Using the above result in Eq. (3.15), the equation of motion of the inflaton field becomes

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \qquad (3.20)$$

Using the definition of the energy-momentum tensor $T_{\mu\nu} \equiv 2\,\delta S/\delta g^{\mu\nu}$ in the Lagrangian formulation of general relativity, we find that the energy-momentum tensor of the inflaton field takes the form

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\Big(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - V(\phi)\Big). \qquad (3.21)$$

Explicitly, the non-zero components of the energy-momentum tensor are

$$T^{\phi}_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \equiv \rho \qquad (3.22)$$
$$T^{\phi}_{ij} = a^2\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\delta_{ij} \equiv a^2\,p\,\delta_{ij}. \qquad (3.23)$$

If we consider the period of time during which the inflation field dominates the universe, we can assume that $T_{\mu\nu} = T^{\phi}_{\mu\nu}$. Since the condition $(\rho + 3p) < 0$ will hold if $\dot{\phi}^2 < V(\phi)$ which is definitely possible, we see that the inflaton model is a good candidate for describing inflation. Using Eqs. (3.10) and (3.22), we can write the Hubble parameter in terms of the inflaton field as

$$H^2 = \frac{8\pi\,G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \qquad (3.24)$$

Taking the time derivative on both sides, we find

$$\begin{aligned} 2H\dot{H} &= \frac{8\pi G}{3}(\dot{\phi}\ddot{\phi} + \dot{\phi}V'(\phi)) \\ &= -8\pi G H\dot{\phi}^2, \\ \dot{H} &= -4\pi G\dot{\phi}^2, \end{aligned} \tag{3.25}$$

where we have used Eq. (3.20). Using the identity $\ddot{a}/a = \dot{H} + H^2$, we see that the universe will exponentially expand during the inflation if $|\dot{H}| \ll H^2$. Since $|\dot{H}| \propto \dot{\phi}^2$ and $H^2 \propto (\dot{\phi}^2 + V(\phi))$, we have the condition $\dot{\phi}^2 \ll V(\phi)$ under which the Hubble parameter can be approximated as

$$H^2 \approx \frac{8\pi G}{3}V(\phi). \tag{3.26}$$

This approximation is called the slow-roll approximation due to the smallness of $\dot{\phi}$. In this approximation, we assume that the inflaton field was originally at the top of the large but flat potential. During the inflation, the inflaton field rolls down very slowly to the vacuum state and the inflation stops when $\dot{\phi}$ is of the same order as the inflaton potential [11, 13]. To keep $\dot{\phi}^2$ small for a long-enough time, we demand that the friction term $3H\dot{\phi}$ in Eq. (3.20) must be large to keep $\dot{\phi}$ close to a constant. This can be achieved if we assume that

$$\ddot{\phi} \ll 3H\dot{\phi}. \tag{3.27}$$

With this condition, Eq. (3.20) can be approximated as

$$3H\dot{\phi} + V'(\phi) \approx 0. \tag{3.28}$$

The condition for the exponentially expanding universe becomes

$$\begin{aligned} \frac{|\dot{H}|}{H^2} &\approx \frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)} \\ &\approx \frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \ll 1 \end{aligned} \tag{3.29}$$

where we have used Eqs. (3.27) and (3.28) to obtain the last line. Also, by taking the time-derivative of Eq. (3.28), we get

$$\begin{aligned} \ddot{\phi} &\approx \frac{d}{dt}\left(-\frac{1}{3H}V'(\phi)\right) \\ &= -\frac{V''(\phi)\dot{\phi}}{3H} + \frac{1}{3H^2}\dot{H}V'(\phi) \\ &= \frac{V'(\phi)}{3}\left(\frac{V''(\phi)}{3H^2} - \frac{1}{16\pi G}\frac{V'^2(\phi)}{V^2(\phi)}\right). \end{aligned} \tag{3.30}$$

As Eqs. (3.27) and (3.28) imply that $\ddot{\phi} \ll V'(\phi)$, the terms in the brackets in Eq. (3.30) must be less than one. The second term is already very small by Eq. (3.29), so that the first term gives us the condition [11]

$$\frac{1}{8\pi G}\left|\frac{V''(\phi)}{V(\phi)}\right| \ll 1. \tag{3.31}$$

Eqs. (3.29) and (3.31) motivated us to define two parameters, ϵ and η, by

$$\epsilon \equiv \frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \tag{3.32}$$

$$\eta \equiv \frac{1}{8\pi G}\left|\frac{V''(\phi)}{V(\phi)}\right|. \tag{3.33}$$

These parameters are called the slow-roll parameters, which satisfy the slow-roll conditions [12, 13],

$$\epsilon \ll 1, \tag{3.34}$$

$$\eta \ll 1, \tag{3.35}$$

in the period of inflation. The condition for ϵ describes the accelerated expanding universe, while the condition for η ensures that such an expansion will last long enough to solve the problems in standard cosmology. Just after the end of inflation, the universe is expected to be cold due to its enormous expansion, and is still dominated by the inflaton field. To recover the Hot Big Bang universe, the energy from the inflaton field must be dissipated out. This stage is called reheating, in which the universe is warmed up and the Standard Model particles are produced [13]. When the inflation ends, we assume that the inflaton field should be close to its own vacuum state, and subsequently undergoes damped oscillation around the vacuum. The particle creation model uses the possibility that the inflaton field may be coupled to other lighter fields. In the reheating stage, the energy from the inflaton field is dissipated to those coupled fields due to damped oscillation, and the particle production begins.

It should be emphasized that the inflaton field was treated as the classical field in all above arguments. In the next section, the inflaton field will be treated as a quantum field, and its equation of motion will be replaced by the dynamical equation for its mean field (which plays the role of ϕ) obtained from the effective action of quantum field theory.

3.2 Nonequilibrium Scalar Field Theory in Curved Spacetime

In the discussion of the time evolution of the inflaton field, it is clear that the system under consideration is time dependent, and therefore must be in a nonequilibrium state. This motivates us to study the nonequilibrium dynamics of a scalar field theory in curved spacetime. In this section, we will therefore derive the dynamical equations for the mean fields and the propagators in a scalar field theory with quartic self-interaction terms in the nonequilibrium setting. Our assumption here is that the system starts out in a pure state, so that the matrix element in Eq. (2.39) is equal to one. The calculation is almost the same as in Chapter 2, except that the background spacetime is now curved. The classical action corresponding to the ϕ^4 theory takes the form [7, 14]

$$S[\phi, g^{\mu\nu}] = S^{\phi}[\phi, g^{\mu\nu}] + S^{G}[g^{\mu\nu}], \tag{3.36}$$

where

$$S^{\phi}[\phi, g^{\mu\nu}] = -\int d^4x\sqrt{-g}\,\frac{1}{2}\left\{\phi(\Box + m^2)\phi + \frac{\lambda}{12}\phi^4\right\} \tag{3.37}$$

$$S^G[g^{\mu\nu}] = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R-2\Lambda) \tag{3.38}$$

with g being the determinant of the metric tensor. Here, R is the scalar curvature and Λ is the cosmological constant. The closed-time path (CTP) action corresponding to the above action takes the form

$$S[\phi_a, g_a^{\mu\nu}] = S[\phi_1, g_1^{\mu\nu}] - S[\phi_2, g_2^{\mu\nu}], \tag{3.39}$$

where

$$S[\phi_i, g_i^{\mu\nu}] = S^\phi[\phi_i, g_i^{\mu\nu}] + S^G[g_i^{\mu\nu}] \tag{3.40}$$

with the index $i = 1, 2$ labeling two time branches. Using this CTP action, we can write down the corresponding CTP-2PI generating functional as

$$\begin{aligned} Z[J^a, K^{ab}, g_a^{\mu\nu}] &= \int D\Phi_a \exp\left[\frac{i}{\hbar}\left\{ S[\Phi_a, g_a^{\mu\nu}] + \int d^4x\sqrt{-g}\,(J^a\Phi_a)\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,(K^{ab}\Phi_a\Phi_b)\right\}\right], \end{aligned} \tag{3.41}$$

where

$$\int d^4x\sqrt{-g}\,J^a\Phi_a = \int d^4x\sqrt{-g_1}\,J_1\Phi_1 - \int d^4x\sqrt{-g_2}\,J_2\Phi_2. \tag{3.42}$$

Note that gravity is treated as a classical background in the above generating functional [14]. Let $W[J^a, K^{ab}, g_a^{\mu\nu}] = -i\hbar\ln Z[J^a, K^{ab}, g_a^{\mu\nu}]$, then the CTP-2PI effective action is defined by

$$\begin{aligned} \Gamma[\phi_a, G_{ab}, g_a^{\mu\nu}] &= W[J^a, K^{ab}, g_a^{\mu\nu}] - \int d^4x\sqrt{-g}\,J^a\phi_a \\ &\quad -\frac{1}{2}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,K^{ab}(\phi_a\phi_b + \hbar G_{ab}), \end{aligned} \tag{3.43}$$

where ϕ_a and G_{ab} are defined as the Legendre transforms

$$\phi_a(x) = \frac{1}{\sqrt{-g}}\frac{\delta W}{\delta J^a(x)} \tag{3.44}$$

$$\hbar G_{ab}(x,x') = 2\,\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta W}{\delta K^{ab}(x,x')} - \phi_a(x)\phi_b(x') \tag{3.45}$$

with the corresponding inverse Legendre transforms

$$\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\phi_a(x)} = -J^a(x) - \int d^4x' \sqrt{-g'}\, K^{ab}(x,x')\phi_b(x') \quad (3.46)$$

$$\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta\Gamma}{\delta G_{ab}(x,x')} = -\frac{\hbar}{2}K^{ab}(x,x'). \quad (3.47)$$

The appearance of the factor $1/\sqrt{-g}$ in front of the above functional differentiation can be explained as follows. We first recall that $\sqrt{-g}\, d^4x$ is an invariant volume element in curved spacetime, so that the identity

$$\int d^4x\sqrt{-g}\,\frac{\delta^4(x-x')}{\sqrt{-g'}} = 1 \quad (3.48)$$

implies that $\delta^4(x-x')/\sqrt{-g'}$ is the covariant Dirac delta function. It follows that $(\sqrt{-g})^{-1}\delta/\delta\phi_a(x)$ is the covariant functional differential operator in the sense that

$$\frac{1}{\sqrt{-g}}\frac{\delta\phi_a(x)}{\delta\phi_b(x')} = \delta_a{}^b\,\frac{\delta^4(x-x')}{\sqrt{-g}} \quad (3.49)$$

is an invariant quantity; this explains the appearance of the factor $1/\sqrt{-g}$ in front of the functional derivative.

Using the background field method, we decompose the field Φ_a as the sum of a mean field ϕ_a and its fluctuation φ_a, that is, $\Phi_a = \phi_a + \varphi_a$, so that the effective action takes the form

$$\begin{aligned}\Gamma[\phi_a, G_{ab}, g_a^{\mu\nu}] = -i\hbar\,\ln\Bigg\{\int D\varphi_a \exp\Bigg[\frac{i}{\hbar}\Bigg(S[\phi_a+\varphi_a, g_a^{\mu\nu}] + \int d^4x\,\sqrt{-g}J^a\varphi_a \\ + \frac{1}{2}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,K^{ab}(\Phi_a\Phi_b - \phi_a\phi_b - \hbar G_{ab})\Bigg)\Bigg]\Bigg\} \\ = -i\hbar\,\ln\Bigg\{\int D\varphi_a \exp\Bigg[\frac{i}{\hbar}\Bigg(S[\phi_a, g_a^{\mu\nu}] + \int d^4x\left(\frac{\delta S}{\delta\phi_a} - \frac{\delta\Gamma}{\delta\phi_a}\right)\varphi_a \\ +\frac{1}{2}\int d^4x\int d^4x'\left(\frac{\delta^2 S}{\delta\phi_a(x)\delta\phi_b(x')} - \frac{2}{\hbar}\frac{\delta\Gamma}{\delta G_{ab}(x,x')}\right)\varphi_a(x)\varphi_b(x') \\ +S_Q + \int d^4x\int d^4x'\frac{\delta\Gamma}{\delta G_{ab}(x,x')}G_{ab}(x,x')\Bigg)\Bigg]\Bigg\}, \quad (3.50)\end{aligned}$$

where S_Q is the part of the action containing the higher-order terms in the field fluctuations. For the ϕ^4 theory under consideration, the explicit form of S_Q is

$$S_Q = -\int d^4x\sqrt{-g}\,\frac{\lambda}{24}c^{abcd}\left\{4\phi_a\varphi_b\varphi_c\varphi_d + \varphi_a\varphi_b\varphi_c\varphi_d\right\}. \quad (3.51)$$

In obtaining the above result, we expanded the action as

$$
\begin{aligned}
S[\phi_a+\varphi_a, g^{\mu\nu}_a] &= S[\phi_a, g^{\mu\nu}_a] + \int d^4x\sqrt{-g}\left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi_a(x)}\right)\varphi_a(x) \\
&+\frac{1}{2}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\left(\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta^2 S}{\delta\phi_b(x)\delta\phi_a(x')}\right)\varphi_b(x)\varphi_a(x') \\
&+S_Q
\end{aligned}
\tag{3.52}
$$

and expressed J^a and K^{ab} in terms of the derivatives of the effective action using Eqs. (3.46) and (3.47). It can be seen that the form of the effective action in Eq. (3.50) is the same as the one obtained in Chapter 2. This leads us to conclude that

$$
\begin{aligned}
\Gamma[\phi_a, G_{ab}, g^{\mu\nu}_a] &= S[\phi_a, g^{\mu\nu}_a] + \frac{\hbar}{2}\int d^4x\int d^4x'\frac{\delta^2 S}{\delta\phi_a(x)\delta\phi_b(x')}G_{ab}(x,x') \\
&-\frac{i\hbar}{2}\ln\det G + \Gamma_2,
\end{aligned}
\tag{3.53}
$$

where Γ_2 is $-i\hbar$ times the sum of 2PI vacuum Feynman diagrams with scalar propagators $\hbar G_{ab}$ and interaction vertices defined by $iS_Q/\hbar$.

The lowest-order contribution to Γ_2 consists of 2-loop diagrams. In the ϕ^4 theory, there are two types of scalar vertices (four-point and three-point vertices) and so there are two types of 2-loop diagrams: a double-bubble diagram and a sunset diagram. The contributions of these diagrams to the effective action are easily found:

$$
\begin{aligned}
\Gamma_2^{double-bubble} &= -i\hbar\left(-\frac{i}{\hbar}\frac{\lambda}{24}\right)\int d^4x\sqrt{-g}\,c^{abcd}(\hbar G_{ab}(x,x))(\hbar G_{cd}(x,x))\frac{4!}{2!2!} \\
&= -\frac{1}{4}\lambda\hbar^2\int d^4x\sqrt{-g}\,c^{abcd}G_{ab}(x,x)G_{cd}(x,x)
\end{aligned}
\tag{3.54}
$$

$$
\begin{aligned}
\Gamma_2^{sunset} &= -i\hbar\left(-\frac{i}{\hbar}\frac{\lambda}{6}\right)^2\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,c^{abcd}c^{a'b'c'd'} \\
&\qquad\times\phi_a(x)\phi_{a'}(x')(\hbar^3 G_{bb'}(x,x')G_{cc'}(x,x')G_{dd'}(x,x'))\frac{3!}{2!} \\
&= \frac{i}{12}\lambda^2\hbar^2\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,c^{abcd}c^{a'b'c'd'} \\
&\qquad\times\phi_a(x)\phi_{a'}(x')G_{bb'}(x,x')G_{cc'}(x,x')G_{dd'}(x,x').
\end{aligned}
\tag{3.55}
$$

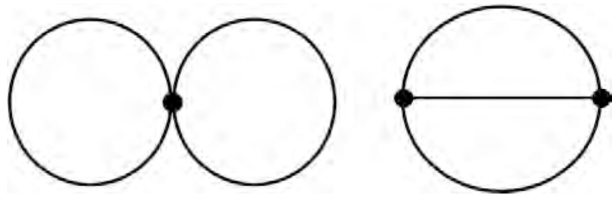

Figure 3.1: The double-bubble and sunset diagrams with the propagators $\hbar G$.

Using Eqs. (3.54) and (3.55), we find that, at 2-loop order, the effective action in Eq. (3.53) becomes

$$\begin{aligned}\Gamma[\phi_a, G_{ab}, g_a^{\mu\nu}] = S[\phi_a, g_a^{\mu\nu}] + \frac{\hbar}{2}\int d^4x \int d^4x' \frac{\delta^2 S}{\delta\phi_b(x')\delta\phi_a(x)} G_{ab}(x,x') \\ -\frac{i\hbar}{2}\ln\det G - \frac{1}{4}\lambda\hbar^2 \int d^4x\sqrt{-g}\, c^{abcd} G_{ab}(x,x) G_{cd}(x,x) \\ +\frac{i}{12}\lambda^2\hbar^2 \int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\, c^{abcd} c^{a'b'c'd'} \\ \times\phi_a(x)\phi_{a'}(x') G_{bb'}(x,x') G_{cc'}(x,x') G_{dd'}(x,x'). \end{aligned} \tag{3.56}$$

The dynamical equations can be obtained by performing the functional differentiation on Γ with respect to $g_a^{\mu\nu}$, ϕ_a and G_{ab} [7]:

$$\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g_a^{\mu\nu}(x)}\bigg|_{\phi_1=\phi_2=\phi, g_1^{\mu\nu}=g_2^{\mu\nu}=g^{\mu\nu}} = 0 \tag{3.57}$$

$$\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\phi_a(x)}\bigg|_{\phi_1=\phi_2=\phi, g_1^{\mu\nu}=g_2^{\mu\nu}=g^{\mu\nu}} = 0 \tag{3.58}$$

$$\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta\Gamma}{\delta G_{ab}(x,x')}\bigg|_{\phi_1=\phi_2=\phi, g_1^{\mu\nu}=g_2^{\mu\nu}=g^{\mu\nu}} = 0. \tag{3.59}$$

Let us first find the dynamical equations for the mean fields. The functional derivatives with respect to ϕ_a of all terms in Eq. (3.56) are as follows:

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi_a(x)} &= -\int d^4x'\sqrt{-g'}\left\{c^{bc}(\Box' + m^2)\phi_c(x')\delta^a{}_b\frac{\delta^4(x'-x)}{\sqrt{-g}}\right. \\ &\quad \left. +\frac{\lambda}{24}c^{bcde}(4)\phi_c(x')\phi_d(x')\phi_e(x')\delta^a{}_b\frac{\delta^4(x'-x)}{\sqrt{-g}}\right\} \\ &= -c^{ab}(\Box + m^2)\phi_b(x) - \frac{\lambda}{6}c^{abcd}\phi_b(x)\phi_c(x)\phi_d(x) \end{aligned} \tag{3.60}$$

$$
\begin{aligned}
&\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi_a(x)}\left(\frac{\hbar}{2}\int d^4x'\sqrt{-g'}\int d^4x''\sqrt{-g''}\right.\\
&\qquad\qquad\qquad\left.\times\left(\frac{1}{\sqrt{-g'}}\frac{1}{\sqrt{-g''}}\frac{\delta^2 S}{\delta\phi_b(x')\delta\phi_c(x'')}\right)G_{bc}(x',x'')\right)\\
&\quad= -\frac{\hbar}{2}\int d^4x'\sqrt{-g'}\int d^4x''\sqrt{-g''}\,c^{bcde}\delta^a{}_d(\lambda\phi_e(x''))\\
&\qquad\qquad\qquad\times\frac{\delta(x''-x')}{\sqrt{-g'}}\frac{\delta(x''-x)}{\sqrt{-g}}G_{bc}(x',x'')\\
&\quad= -\frac{\hbar}{2}\lambda c^{bcae}\phi_e(x)G_{bc}(x,x)
\end{aligned}
\tag{3.61}
$$

$$
\begin{aligned}
&\frac{1}{\sqrt{-g}}\frac{\delta\Gamma_2}{\delta\phi_a(x)}\\
&\quad= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi_a(x)}\left\{\frac{i}{12}\lambda^2\hbar^2\int d^4x'\sqrt{-g'}\int d^4x''\sqrt{-g''}\,c^{a'b'c'd'}c^{a''b''c''d''}\right.\\
&\qquad\qquad\left.\times\phi_{a'}(x')\phi_{a''}(x'')G_{b'b''}(x',x'')G_{c'c''}(x',x'')G_{d'd''}(x',x'')\right\}\\
&\quad= \frac{i}{6}\lambda^2\hbar^2\int d^4x'\sqrt{-g'}\int d^4x''\sqrt{-g''}\,c^{a'b'c'd'}c^{a''b''c''d''}\\
&\qquad\qquad\times\frac{\delta^a{}_{a'}\delta(x'-x)}{\sqrt{-g}}\phi_{a''}(x'')G_{b'b''}(x',x'')G_{c'c''}(x',x'')G_{d'd''}(x',x'')\\
&\quad= \frac{i}{6}\lambda^2\hbar^2\int d^4x'\sqrt{-g'}\,c^{abcd}c^{a'b'c'd'}\phi_{a'}(x')G_{bb'}(x,x')G_{cc'}(x,x')G_{dd'}(x,x').
\end{aligned}
\tag{3.62}
$$

Using the above result, we get

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\phi_a(x)}\\
&= \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi_a(x)}+\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi_a(x)}\left(\frac{\hbar}{2}\int d^4x'\int d^4x''\frac{\delta^2 S}{\delta\phi_{b'}\delta\phi_{c''}}G_{b'c''}\right)\\
&\quad+\frac{1}{\sqrt{-g}}\frac{\delta\Gamma_2}{\delta\phi_a(x)},
\end{aligned}
\tag{3.63}
$$

which results in the dynamical equations for the mean fields:

$$
\begin{aligned}
&\left\{c^{ab}(\Box+m^2)+\frac{\lambda}{6}c^{abcd}\phi_c(x)\phi_d(x)+\frac{\hbar}{2}\lambda c^{cdab}G_{cd}(x,x)\right\}\phi_b(x)\\
&\qquad\qquad\qquad-\int d^4x'\sqrt{-g'}\,\Sigma(x,x')^{aa'}\phi_{a'}(x')=0,
\end{aligned}
\tag{3.64}
$$

where the nonlocal function $\Sigma(x,x')$ is defined as

$$
\Sigma(x,x')^{aa'}=\frac{i}{6}\lambda^2\hbar^2c^{abcd}c^{a'b'c'd'}G_{bb'}(x,x')G_{cc'}(x,x')G_{dd'}(x,x'). \tag{3.65}
$$

We next turn to the dynamical equations for the propagators:

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta\Gamma}{\delta G_{ab}(x,x')} \\
&= \frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta}{\delta G_{ab}(x,x')}\Big\{\frac{\hbar}{2}\int d^4y\int d^4y'\frac{\delta^2 S}{\delta\phi_{b'}(y)\delta\phi_{a'}(y')}G_{a'b'}(y,y') \\
&\qquad -\frac{i\hbar}{2}\ln\det G+\Gamma_2\Big\}. \qquad (3.66)
\end{aligned}
$$

Using

$$
\frac{\delta G_{ab}(x,x')}{\delta G_{cd}(y,y')}=\delta_a{}^c\delta_b{}^d\delta^4(x-y)\delta^4(x'-y'), \qquad (3.67)
$$

each term on the right-hand side can be evaluated as follows:

$$
\begin{aligned}
&\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta}{\delta G_{ab}(x,x')}\left\{\frac{\hbar}{2}\int d^4y\int d^4y'\frac{\delta^2 S}{\delta\phi_c(y)\delta\phi_d(y')}G_{cd}(y,y')\right\} \\
&= \frac{\hbar}{2}\int d^4y\sqrt{-g_y}\int d^4y'\sqrt{-g'_y} \\
&\qquad \times\left(\frac{1}{\sqrt{-g_y}}\frac{1}{\sqrt{-g'_y}}\frac{\delta^2 S}{\delta\phi_c(y)\delta\phi_d(y')}\right)\frac{\delta(y-x)}{\sqrt{-g}}\frac{\delta(y'-x')}{\sqrt{-g'}}\delta^a{}_c\delta^b{}_d \\
&= \frac{\hbar}{2}\left(\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta^2 S}{\delta\phi_a(x)\delta\phi_b(x')}\right) \\
&= -\frac{\hbar}{2}\Big(c^{ab}(\Box+m^2)+\frac{\lambda}{2}c^{abcd}\phi_c(x)\phi_d(x)\Big)\frac{\delta^4(x-x')}{\sqrt{-g'}} \qquad (3.68)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{i\hbar}{2}\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta}{\delta G_{ab}(x,x')}(\ln\det G) \\
&\qquad = -\frac{i\hbar}{2}\frac{1}{\det G}\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta}{\delta G_{ab}(x,x')}(\det G) \\
&\qquad = -\frac{i\hbar}{2}G^{-1}_{ab}(x,x') \qquad (3.69)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta\Gamma_2}{\delta G_{ab}(x,x')} \\
&= \frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta}{\delta G_{ab}(x,x')}\Big\{-\frac{1}{4}\lambda\hbar^2\int d^4y\sqrt{-g_y}\,c^{a'b'c'd'}G_{a'b'}(y,y)G_{c'd'}(y,y) \\
&\qquad +\frac{i}{12}\lambda^2\hbar^2\int d^4y\sqrt{-g_y}\int d^4y'\sqrt{-g'_y}\,c^{a'b'c'd'}c^{a''b''c''d''} \\
&\qquad\qquad \times\phi_{a'}(y)\phi_{a''}(y')G_{b'b''}(y,y')G_{c'c''}(y,y')G_{d'd''}(y,y')\Big\} \\
&= -\frac{1}{2}\lambda\hbar^2\int d^4y\sqrt{-g_y}\,c^{a'b'c'd'}G_{c'd'}(y,y)\frac{\delta^4(y-x)}{\sqrt{-g}}\frac{\delta^4(y-x')}{\sqrt{-g'}}\delta^a{}_{a'}\delta^b{}_{b'}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{i}{4}\lambda^2\hbar^2\int d^4y\sqrt{-g_y}\int d^4y'\sqrt{-g'_y}\,c^{a'b'c'd'}\,c^{a''b''c''d''}\\
&\qquad\times\phi_{a'}(y)\phi_{a''}(y')G_{c'c''}(y,y')G_{d'd''}(y,y')\frac{\delta^4(y-x)}{\sqrt{-g}}\frac{\delta^4(y'-x')}{\sqrt{-g'}}\delta^a{}_{b'}\delta^b{}_{b''}\\
= &-\frac{1}{2}\lambda\hbar^2c^{abcd}G_{cd}(x,x)\frac{\delta^4(x-x')}{\sqrt{-g'}}\\
&+\frac{i}{4}\lambda^2\hbar^2c^{cade}c^{c'bd'e'}\phi_c(x)\phi_{c'}(x')G_{dd'}(x,x')G_{ee'}(x,x'). \qquad (3.70)
\end{aligned}
$$

Substituting Eqs. (3.68)–(3.70) into Eq. (3.66), we obtain

$$
\begin{aligned}
iG^{-1}_{ab}(x,x') = &-\Big(c^{ab}(\Box+m^2)+\frac{\lambda}{2}c^{abcd}\phi_c\phi_d+\lambda\hbar c^{abcd}G_{cd}(x,x)\Big)\frac{\delta^4(x-x')}{\sqrt{-g'}}\\
&+\frac{i\lambda^2}{2}\hbar c^{cade}c^{c'bd'e'}\phi_c(x)\phi_{c'}(x')G_{dd'}(x,x')G_{ee'}(x,x'). \qquad (3.71)
\end{aligned}
$$

Multiplying both sides by $\sqrt{-g'}\,G_{bf}(x',x'')$ and performing an integration over x' using the formula

$$
\int d^4x'\sqrt{-g'}\,G^{-1}_{ab}(x,x')G_{bf}(x',x'') = \delta_{af}\frac{\delta^4(x-x'')}{\sqrt{-g}}, \qquad (3.72)
$$

we obtain the dynamical equations for the propagators:

$$
\begin{aligned}
&\Big\{c^{ac}(\Box+m^2)+\frac{1}{2}\lambda c^{acde}\phi_d(x)\phi_e(x)+\lambda\hbar c^{acde}G_{de}(x,x)\Big\}G_{cb}(x,x')\\
&\quad-\frac{i}{2}\lambda^2\hbar\int d^4x''\sqrt{-g'}\,c^{adef}c^{cd'e'f'}\phi_d(x'')\phi_{d'}(x)G_{ee'}(x'',x)G_{ff'}(x'',x)G_{cb}(x'',x')\\
&\quad=-i\delta_b{}^a\frac{\delta^4(x-x')}{\sqrt{-g}}. \qquad (3.73)
\end{aligned}
$$

Observe that the dynamical equations in Eqs. (3.64) and (3.73) are coupled equations of the mean fields and the propagators, whose solutions are extremely hard to obtain analytically.

Finally, let us consider the functional derivative of the effective action with respect to the metric tensor:

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g_a^{\mu\nu}(x)} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_a^{\mu\nu}(x)}(\Gamma_T+S^G)\\
&= \frac{1}{\sqrt{-g}}\frac{\delta S^G}{\delta g_a^{\mu\nu}(x)}+\frac{1}{2}\langle T_{\mu\nu}\rangle, \qquad (3.74)
\end{aligned}
$$

where we have defined $\Gamma_T[\phi_a, G_{ab}, g_a^{\mu\nu}] \equiv \Gamma[\phi_a, G_{ab}, g_a^{\mu\nu}] - S^G[g_a^{\mu\nu}]$ and

$$\langle T_{\mu\nu} \rangle \equiv \frac{2}{\sqrt{-g}} \frac{\delta \Gamma_T}{\delta g_a^{\mu\nu}}. \tag{3.75}$$

Using the gravitational action in Eq. (3.38), we find

$$\frac{1}{\sqrt{-g}} \frac{\delta S^G}{\delta g_a^{\mu\nu}} = -\frac{1}{16\pi G} \left(G_{\mu\nu} + \Lambda g_{\mu\nu} \right), \tag{3.76}$$

where G and $G_{\mu\nu}$ are, respectively, the Newtonian gravitational constant and the Einstein tensor, so that Eq. (3.74) becomes

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi\, G \langle T_{\mu\nu} \rangle \tag{3.77}$$

which is the Einstein equation (with the cosmological constant), provided that we identify $\langle T_{\mu\nu} \rangle$ as the classical energy-momentum tensor. Thus, Eq. (3.75) enables us to calculate the expectation value of the energy-momentum tensor with quantum corrections included.

3.3 Nonequilibrium Inflaton Dynamics in Friedmann-Robertson-Walker Spacetime

Having obtained the dynamical equations for the mean fields and the propagators in the previous section, we now consider a specific case of the Friedmann-Robertson-Walker (FRW) spacetime. Keeping the terms up to the first order in λ in Eq. (3.64), the dynamical equation for the mean field reads

$$\left\{ (\Box + m^2) + \frac{\lambda}{6}\phi(x)^2 + \frac{\lambda}{2}\hbar G(x,x) \right\} \phi(x) = 0, \tag{3.78}$$

where we have taken the coincidence limit in which $\phi_1 = \phi_2 \equiv \phi$ and $G(x,x) \equiv G_{11}(x,x)|_{\phi_1=\phi_2=\phi}$. Using the FRW metric presented in Section 3.1, the above equation takes the form [7]

$$\ddot{\phi}(x) + 3H\dot{\phi}(x) + \left\{ m^2 + \frac{\lambda}{6}\phi^2(x) + \frac{\lambda}{2}\hbar G(x,x) \right\} \phi(x) = 0. \tag{3.79}$$

The terms in the curly brackets can be interpreted as the potential terms for the inflaton. By comparing the above result with Eq. (3.20), we see that, with the quantum corrections taken into account, the inflaton potential depends on $G(x,x)$ apart from the self-interaction terms and the spacetime curvature effect. Recall that $\hbar G$ is the propagator of the fluctuation field φ, we find

$$\begin{aligned} \hbar G(x,x) \equiv \hbar G_{11}(x,x) &= \langle \varphi(x)\varphi(x) \rangle \\ &= \langle \varphi^2(x) \rangle, \end{aligned} \tag{3.80}$$

so that $G(x,x)$ is the variance of the fluctuation field [3, 7]. Note that, since the two fluctuation fields in $G(x,x)$ take their values at the same spacetime point, there is no need to write down the time-ordering operator.

Even though $G(x,x')$ is formally the fluctuation-field propagator which should be computed using the knowledge of the fluctuation fields and their interactions, in the 2PI formalism, $G(x,x')$ is treated as an independent quantity satisfying its own dynamical equation. Using the FRW metric and keeping the terms up to the first order in λ, Eq. (3.73) becomes

$$\begin{aligned} -\frac{i}{a^3}\delta^4(x-x') &= \Big\{ \frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} - \frac{1}{a^2}\nabla^2 + m^2 \\ &\quad + \frac{\lambda}{2}\phi^2(x) + \lambda\hbar G(x,x) \Big\} G(x,x'), \end{aligned} \tag{3.81}$$

where the coincidence limit has been taken. As the term $G(x,x)$ in Eqs. (3.79) and (3.81) is multiplied by $\hbar$, this term therefore provides the first-order quantum correction to the dynamical equations. Eqs. (3.79) and (3.81) constitute a system of second-order partial differential equations, from which the mean field and the propagator can be determined. Solving these equations, however, is a formidable task and people normally employ the numerical techniques to attack this problem.

CHAPTER IV

NONEQUILIBRIUM DYNAMICS OF THE INFLATON WITH FERMION COUPLING

In the previous chapter, we have investigated the nonequilibrium scalar ϕ^4 theory in a classical gravitational background, and mentioned that this theory should describe the nonequilibrium dynamics of the inflaton field. After the end of inflation, the stage of reheating begins, and the inflaton energy is dissipated out to creating the Standard Model particles via the coupling of the inflaton field with other fields. This motivates us to study the nonequilibrium dynamics of a scalar field coupled to a fermion field. In this chapter, we will derive the dynamical equations for the mean fields and the propagators of the scalar ϕ^4 theory coupled to a fermion via the Yukawa interaction, and analyze the causality of the dynamical equations. Our calculation will be limited to a 1-loop level of quantum calculations.

4.1 The Model and the Coarse-Grained Effective Action

Let us add a fermion to the scalar-field model considered in the previous chapter such that its interaction with the scalar fields is of the form of the Yukawa interaction [8]. The resulting action thus consists of the scalar-field ϕ^4 action, the Dirac action, the Yukawa action term, and the gravity action. The corresponding CTP action takes the form

$$S[\Phi_a, \Psi_a, \overline{\Psi}_a, g_a^{\mu\nu}] \equiv S[\Phi_+, \Psi_+, \overline{\Psi}_+, g_+^{\mu\nu}] - S[\Phi_-, \Psi_-, \overline{\Psi}_-, g_-^{\mu\nu}], \tag{4.1}$$

where

$$S[\Phi_a, \Psi_a, \overline{\Psi}_a, g_a^{\mu\nu}] = S^\phi[\Phi_a, g_a^{\mu\nu}] + S^\Psi[\Psi_a, \overline{\Psi}_a, g_a^{\mu\nu}] + S^G[g_a^{\mu\nu}] + S^Y[\Phi_a, \Psi_a, \overline{\Psi}_a, g_a^{\mu\nu}], \tag{4.2}$$

with the index $a = 1, 2$ being the time-branch index, and S^{ϕ}, S^{Ψ}, S^{G}, and S^{Y} being the ϕ^4 scalar-field action, the Dirac action, the gravity action, and the Yukawa interaction, respectively. As we will not consider the functional derivative of the effective action with respect to the metric tensor, it is legitimate to set the metric tensors on two time branches to be equal, that is, we set $g_{+}^{\mu\nu} = g_{-}^{\mu\nu} = g^{\mu\nu}$ in the above action. With this simplification, the two gravity actions in the CTP action cancel each other, and so we will not be dealing with the gravity action any more. The explicit forms of the other three parts of the action are

$$\begin{aligned} S^{\phi}[\Phi_a, g^{\mu\nu}] &= -\frac{1}{2}\int d^4x\sqrt{-g}\,\Big(c^{ab}\Phi_a(x)(\Box + m^2)\Phi_b(x) \\ &\qquad + \frac{\lambda}{12}c^{abcd}\Phi_a(x)\Phi_b(x)\Phi_c(x)\Phi_d(x)\Big) \qquad (4.3) \\ S^{\Psi}[\Psi_a, \overline{\Psi}_a, g^{\mu\nu}] &= \int d^4x\sqrt{-g}\,\Big\{\frac{i}{2}c^{ab}\Big(\overline{\Psi}_a(x)\gamma^{\mu}\nabla_{\mu}\Psi_b(x) \\ &\qquad -(\nabla_{\mu}\overline{\Psi}_a(x))\gamma^{\mu}\Psi_b(x) - \mu\overline{\Psi}_a(x)\Psi_b(x)\Big)\Big\} \qquad (4.4) \\ S^{Y}[\Phi_a, \Psi_a, \overline{\Psi}_a, g^{\mu\nu}] &= -f\int d^4x\sqrt{-g}\,\Big(c^{abc}\Phi_a(x)\overline{\Psi}_b(x)\Psi_a(x)\Big). \qquad (4.5) \end{aligned}$$

The corresponding CTP-2PI generating functional is

$$\begin{aligned} Z[J^a, K^{ab}, g^{\mu\nu}] &= \int D\Phi_a D\Psi_a D\overline{\Psi}_a \exp\Big\{\frac{i}{\hbar}\Big[S[\Phi_a, \Psi_a, \overline{\Psi}_a, g^{\mu\nu}] \\ &+ \int d^4x\sqrt{-g}\,J^a(x)\Phi_a(x) \\ &+ \frac{1}{2}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,K^{ab}(x,x')\Phi_a(x)\Phi_b(x')\Big]\Big\} \qquad (4.6) \end{aligned}$$

and, with $W = -i\hbar\ln Z$, the effective action is defined by

$$\begin{aligned} \Gamma[\phi_a, G_{ab}, g^{\mu\nu}] = W[J^a, K^{ab}, g^{\mu\nu}] - \int d^4x\sqrt{-g}\,J^a(x)\phi_a(x) \\ -\frac{1}{2}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,K^{ab}(x,x')\{\phi_a(x)\phi_b(x') + \hbar G_{ab}(x,x')\}, \qquad (4.7) \end{aligned}$$

where the mean fields ϕ_a and the propagators G_{ab} are defined by the Legendre transforms,

$$\begin{aligned} \frac{1}{\sqrt{-g}}\frac{\delta W}{\delta J^a(x)} &= \phi_a(x) \qquad (4.8) \\ \frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta W}{\delta K^{ab}(x,x')} &= \frac{1}{2}\Big(\phi_a(x)\phi_b(x') + \hbar G_{ab}(x,x')\Big), \qquad (4.9) \end{aligned}$$

with the corresponding inverse transforms,

$$\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\phi_a(x)} = -J^a(x) - \int d^4x'\sqrt{-g'}\,K^{ab}(x,x')\phi_b(x') \quad (4.10)$$

$$\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta\Gamma}{\delta G_{ab}(x,x')} = -\frac{\hbar}{2}K^{ab}(x,x'). \quad (4.11)$$

In the above definition of the effective action, one should observe that we performed the Legendre transforms with respect to the scalar fields, but not the spinor field. The reason for this is that the fermion is treated as the environment of the scalar fields, which will be integrated out when evaluating the generating functional [10]. As a result, the 2PI part of the effecting action (to be obtained below) is 2-particle-irreducible with respect to the scalar cuts only, and remains 1-particle-irreducible with respect to the fermion cuts.[1] Thus, the above effective action is called the "CTP-2PI coarse-grained effective action" [8], in contrast with the fully 2-particle-irreducible effective action in which we perform the Legendre transforms with respect to both scalar and fermion fields.

Using the background field method in which we express the scalar field Φ_a as a sum of the mean field ϕ_a and its fluctuation φ_a, the effective action reads

$$\exp\left\{\frac{i}{\hbar}\Gamma[\phi_a, G_{ab'}, g^{\mu\nu}]\right\} = \int D\Phi_a D\Psi_a D\overline{\Psi}_a \exp\left\{\frac{i}{\hbar}\Big[S[\Phi_a,\Psi_a,\overline{\Psi}_a,g^{\mu\nu}] + \int d^4x\sqrt{-g}\,J^a(x)(\Phi_a(x)-\phi_a(x)) + \frac{1}{2}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,K^{ab}(x,x')\{\Phi_a(x)\Phi_b(x') - \phi_a(x)\phi_b(x') - \hbar G_{ab}(x,x')\}\Big]\right\}. \quad (4.12)$$

The ϕ^4 scalar-field part of the action is the same as in the previous chapter, while the rest of the action takes the explicit form

$$S^\Psi[\Psi_a,\overline{\Psi}_a,g^{\mu\nu}] + S^Y[\phi_a+\varphi_a,\Psi_a,\overline{\Psi}_a,g^{\mu\nu}] = S^\Psi[\phi_a,\Psi_a,\overline{\Psi}_a,g^{\mu\nu}] + S^Y[\varphi_a,\Psi_a,\overline{\Psi}_a,g^{\mu\nu}], \quad (4.13)$$

[1] What this really means is that the corresponding Feynman diagrams cannot be separated into two parts by cutting two internal lines, where one of these lines must be a scalar-field propagator while the other line can be the propagator of any kind.

where

$$\begin{aligned} S^{\Psi}[\phi_a, \Psi_a, \overline{\Psi}_a, g^{\mu\nu}] &= \int d^4x\sqrt{-g}\,\Big(\frac{i}{2}c^{ab}\{\overline{\Psi}_a(x)\gamma^\mu\nabla_\mu\Psi_b(x) \\ &\quad -(\nabla_\mu\overline{\Psi}_a(x))\gamma^\mu\Psi_b(x)\} \\ &\quad -\{c^{ab}\mu + c^{abc}f\phi_c(x)\}\overline{\Psi}_a(x)\Psi_b(x)\Big) \qquad (4.14) \\ S^{Y}[\varphi_a, \Psi_a, \overline{\Psi}_a, g^{\mu\nu}] &= -f\int d^4x\sqrt{-g}\,c^{abc}\varphi_a(x)\overline{\Psi}_b(x)\Psi_c(x). \qquad (4.15) \end{aligned}$$

Using Eq. (3.52) to expand the scalar-field part of the action, and using Eqs. (4.10) and (4.11) to express J^a and K^{ab} in terms of the functional derivatives of Γ, Eq. (4.12) becomes

$$\Gamma[\phi_a, G_{ab}, g^{\mu\nu}] = S^{\phi}[\phi_a, g^{\mu\nu}] + \frac{\delta\Gamma}{\delta G_{ab}}G_{ab} + \Gamma_1 \qquad (4.16)$$

where

$$\begin{aligned} \Gamma_1 &= -i\hbar\,\ln\Big\{\int D\varphi_a D\Psi_a D\bar{\Psi}_a \exp\Big[\frac{i}{\hbar}\Big(S^{\Psi}[\phi_a, \Psi_a, \bar{\Psi}_a, g^{\mu\nu}] \\ &\quad +S^{Y}[\varphi_a, \Psi_a, \bar{\Psi}_a, g^{\mu\nu}] + \int d^4x\left(\frac{\delta S^{\phi}}{\delta\phi_a(x)} - \frac{\delta\Gamma}{\delta\phi_a(x)}\right)\varphi_a(x) \\ &\quad +\frac{1}{2}\int d^4x\int d^4x'\left(\frac{\delta^2 S^{\phi}}{\delta\phi_b(x')\delta\phi_a(x)} - \frac{2}{\hbar}\frac{\delta\Gamma}{\delta G_{ab}(x,x')}\right)\varphi_a(x)\varphi_b(x') \\ &\quad +S_Q\Big)\Big]\Big\}, \qquad (4.17) \end{aligned}$$

with S_Q being the part of the action containing the higher-order terms in the scalar-field fluctuations. To get the 2PI part from the above effective action, let us define a nonlocal kernel $\tilde{K}^{ab}(x,x')$ by (see Chapter 2)

$$\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\left(\frac{\delta^2 S^{\phi}}{\delta\phi_a(x)\delta\phi_b(x')} - \frac{2}{\hbar}\frac{\delta\Gamma}{\delta G_{ab}(x,x')}\right) = iG^{-1}_{ab}(x,x') + \tilde{K}^{ab}(x,x'), \qquad (4.18)$$

where $G^{-1}_{ab}(x,x')/\hbar$ is the inverse of the propagator $\hbar G_{ab}(x,x')$, and let

$$i(F^{-1}_{ab})_{\alpha\beta}(x,x') \equiv \frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\left(\frac{\delta^2 S^{\Psi}[\phi, \Psi, \overline{\Psi}, g^{\mu\nu}]}{\delta\Psi_{b\beta}(x')\delta\overline{\Psi}_{a\alpha}(x)}\right) \qquad (4.19)$$

be the inverse of the fermion propagator, where α and β are spinor indices. (Note that we normally think of F^{-1} as a matrix whose index is represented by a pair of indices $(a\alpha)$, with a being the time-branch index and α the spinor index.) We find

$$\begin{aligned}\Gamma[\phi_a, G_{ab}, g^{\mu\nu}] &= S^{\phi}[\phi_a, g^{\mu\nu}] + \int d^4x \int d^4x' \, \frac{\hbar}{2}\left(\frac{\delta^2 S^{\phi}}{\delta\phi_a(x)\delta\phi_b(x')}\right) G_{ab}(x,x') \\ &\quad -\frac{i\hbar}{2}\ln\det G + i\hbar \ln\det F + \Gamma_2[\phi_a, G_{ab}, g^{\mu\nu}] + const. \quad (4.20)\end{aligned}$$

where

$$\begin{aligned}\Gamma_2[\phi_a, G_{ab}, g^{\mu\nu}] &= -i\hbar \ln\left(\frac{\det(\hbar F)}{\det^{1/2}(\hbar G)}\int D\varphi_a D\Psi_a D\overline{\Psi}_a \exp\Big\{\right. \\ &\quad -\frac{1}{2}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,(\hbar G_{ab})^{-1}(x,x')\varphi_a(x)\varphi_b(x') \\ &\quad -\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,\overline{\Psi}_a(x)(\hbar F_{ab})^{-1}(x,x')\Psi_b(x') \\ &\quad +\frac{i}{\hbar}\Big(S_Q + S^Y[\varphi_a, \Psi_a, \overline{\Psi}_a] + \int d^4x\sqrt{-g}\,\tilde{J}^a(x)\varphi_a(x) \\ &\quad +\frac{1}{2}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,\tilde{K}^{ab}(x,x')\{\varphi_a(x)\varphi_b(x') \\ &\quad \left. -\hbar G_{ab}(x,x')\}\Big)\Big\}\right), \quad (4.21)\end{aligned}$$

with

$$\begin{aligned}\tilde{J}^a(x) &= -\frac{1}{\sqrt{-g}}\Big(\frac{\hbar}{2}\int d^4x'\int d^4x''\frac{\delta^3 S^{\phi}}{\delta\phi_a(x)\delta\phi_b(x')\delta\phi_c(x'')}G_{bc}(x',x'') \\ &\quad +\frac{\delta\Gamma_2}{\delta\phi_a(x)}\Big) \quad (4.22)\\ \tilde{K}^{ab}(x,x') &= -\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{2}{\hbar}\frac{\delta\Gamma_2}{\delta G_{ab}(x,x')}. \quad (4.23)\end{aligned}$$

The explicit forms of S_Q and S^Y in Eq. (4.21) are

$$\begin{aligned}S_Q &= \int d^4x\sqrt{-g}\,c^{abcd}\left(-\frac{\lambda}{24}\varphi_a(x)\varphi_b(x)\varphi_c(x)\varphi_d(x)\right. \\ &\quad \left. -\frac{\lambda}{6}\phi_a(x)\varphi_b(x)\varphi_c(x)\varphi_d(x)\right) \quad (4.24)\\ S^Y &= \int d^4x\sqrt{-g}\,c^{abc}\left(-f\varphi_a(x)\overline{\Psi}_b(x)\Psi_c(x)\right). \quad (4.25)\end{aligned}$$

The derivation of the above result is as follows. From Eq. (4.18), we find

$$\frac{\delta\Gamma}{\delta G_{ab}}G_{ab} = \frac{\hbar}{2}\frac{\delta^2 S^{\phi}}{\delta\phi_a\delta\phi_b}G_{ab} - \frac{\hbar}{2}\sqrt{-g}\sqrt{-g'}\tilde{K}^{ab}G_{ab} + const., \quad (4.26)$$

where the constant term is proportional to

$$\sqrt{-g}\sqrt{-g'}G^{-1}_{ab}G_{ab} \equiv \int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,G^{-1}_{ab}(x,x')G_{ab}(x,x') = 2. \quad (4.27)$$

From Eq. (4.20), by performing the functional differentiation on Γ with respect to $\phi_a(x)$ and $G_{ab}(x,x')$, we obtain

$$\begin{aligned}
&\left(\frac{\delta S^\phi}{\delta\phi_a(x)} - \frac{\delta\Gamma}{\delta\phi_a(x)}\right)\\
&\quad = -\int d^4x'\int d^4x''\,\frac{\hbar}{2}\left(\frac{\delta^3 S^\phi}{\delta\phi_a(x)\delta\phi_b(x')\delta\phi_c(x'')}\right)G_{bc}(x',x'')\\
&\quad -\frac{\delta\Gamma_2}{\delta\phi_a(x)} && (4.28)\\
&\frac{1}{\sqrt{-g}\sqrt{-g'}}\left(\frac{\delta^2 S^\phi}{\delta\phi_a(x)\delta\phi_b(x')} - \frac{2}{\hbar}\frac{\delta\Gamma}{\delta G_{ab}(x,x')}\right)\\
&\quad = iG^{-1}_{ab}(x,x') - \frac{1}{\sqrt{-g}\sqrt{-g'}}\frac{2}{\hbar}\frac{\delta\Gamma_2}{\delta G_{ab}(x,x')}. && (4.29)
\end{aligned}$$

Using Eqs. (4.18), (4.26), and (4.28) in Eq. (4.17), and comparing Eq. (4.29) with Eq. (4.18), we obtain Eqs. (4.20)–(4.23).

With the form of the effective action in Eq. (4.20), we conclude that Γ_2 is the 2PI effective action with respect to the scalar cuts, and is equal to $-i\hbar$ times the sum of 2-particle irreducible diagrams (irreducible with respect to the scalar propagators, not the fermion propagators). Let us now calculate the lowest-order contribution to Γ_2. At 2-loop order, there are three diagrams which contribute to Γ_2: a double-bubble diagram, a sunset diagram and the diagram in Fig. 4.1. The first two diagrams have been calculated in Eqs. (3.54) and (3.55), and the diagram in Fig. 4.1 contributes a term

$$\begin{aligned}
\Gamma_2^Y &= -i\hbar\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,\frac{1}{2}c^{abc}c^{a'b'c'}(-1)\left(-\frac{if}{\hbar}\right)^2\left(\hbar^3 G_{aa'}\mathrm{Tr}\left\{F_{bb'}F_{c'c}\right\}\right)\\
&= -\frac{i}{2}\hbar^2 f^2\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,c^{abc}c^{a'b'c'}G_{aa'}(x,x')\\
&\qquad\qquad \times \mathrm{Tr}\{F_{bb'}(x,x')F_{c'c}(x',x)\} \qquad (4.30)
\end{aligned}$$

to Γ_2, where the trace on the right-hand side is taken over spinor indices. Keeping only the terms of the first order in λ, we discard the sunset diagram since it is of the order λ^2. Thus, to the order λ, only the double-bubble diagram and the diagram in Fig. 4.1 contribute to Γ_2, so that

$$\begin{aligned}\Gamma_2 &= -\frac{\lambda\hbar^2}{4}\int d^4x\sqrt{-g}\,c^{abcd}G_{ab}(x,x)G_{cd}(x,x)\\ &\quad -\frac{i}{2}f^2\hbar^2\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,c^{abc}c^{a'b'c'}G_{aa'}(x,x')\\ &\qquad\qquad \times\,\mathrm{Tr}\{F_{bb'}(x,x')F_{c'c}(x',x)\}. \end{aligned}\tag{4.31}$$

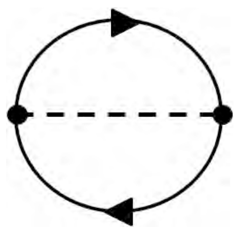

Figure 4.1: A 2-loop diagram with two fermion propagators and a scalar-field propagator.

Having found the effective action, we are now ready to derive the dynamical equations. Begin with the dynamical equations for the mean fields, we need to calculate $\delta\Gamma/\delta\phi_a$:

$$\begin{aligned}0 &= \frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\phi_a(x)}\\ &= \frac{1}{\sqrt{-g}}\frac{\delta S^{\phi}}{\delta\phi_a(x)}+\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi_a(x)}\left(\frac{\hbar}{2}\int d^4x'\int d^4x''\frac{\delta^2 S^{\phi}}{\delta\phi_b(x')\delta\phi_c(x'')}G_{bc}(x',x'')\right)\\ &\quad +i\hbar\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi_a(x)}(\ln\det F)+\frac{1}{\sqrt{-g}}\frac{\delta\Gamma_2}{\delta\phi_a(x)}. \end{aligned}\tag{4.32}$$

The first two terms on the right-hand side of Eq. (4.32) can be evaluated as follows:

$$\frac{1}{\sqrt{-g}}\frac{\delta S^{\phi}}{\delta\phi_a(x)} = -c^{ab}(\Box+m^2)\phi_b(x)-\frac{\lambda}{6}c^{abcd}\phi_b(x)\phi_c(x)\phi_d(x)\tag{4.33}$$

$$\begin{aligned}&\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi_a(x)}\left(\frac{\hbar}{2}\int d^4x'\int d^4x''\left(\frac{\delta^2 S^{\phi}}{\delta\phi_b(x')\delta\phi_c(x'')}\right)G_{bc}(x'x'')\right)\\ &\qquad\qquad = -\frac{\hbar}{2}\lambda c^{bcae}\phi_e(x)G_{bc}(x,x). \end{aligned}\tag{4.34}$$

The third term in Eq. (4.32) can be expressed as

$$\begin{aligned} i\hbar\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi_a(x)}(\ln\,\det F) &= -i\hbar\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi_a(x)}(\ln\,\det F^{-1}) \\ &= -i\hbar\int d^4x'\sqrt{-g'}\int d^4x''\sqrt{-g''}\,\frac{1}{\sqrt{-g}}\frac{\delta(F^{-1}_{bc})_{\beta\gamma}(x',x'')}{\delta\phi_a(x)} \\ &\quad\times\frac{1}{\sqrt{-g'}}\frac{1}{\sqrt{-g''}}\frac{\delta\ln\,\det F^{-1}}{\delta(F^{-1}_{bc})_{\beta\gamma}(x',x'')}. \end{aligned} \tag{4.35}$$

Using the explicit form of $(F^{-1}_{ab})_{\alpha\beta}(x,x')$,

$$i(F^{-1}_{ab})_{\alpha\beta}(x,x') = \frac{\delta^4(x-x')}{\sqrt{-g'}}\{c^{ab}(i\gamma^\mu_{\alpha\beta}\nabla_\mu - \mu\delta_{\alpha\beta}) - c^{abc}f\phi_c(x)\delta_{\alpha\beta}\}, \tag{4.36}$$

we can calculate

$$\frac{1}{\sqrt{-g}}\frac{\delta(F^{-1}_{bc})_{\beta\gamma}(x',x'')}{\delta\phi_a(x)} = ic^{bca}\delta_{\beta\gamma}\,f\frac{\delta^4(x'-x'')}{\sqrt{-g''}}\frac{\delta^4(x'-x)}{\sqrt{-g}}, \tag{4.37}$$

and

$$\frac{1}{\sqrt{-g'}}\frac{1}{\sqrt{-g''}}\frac{\delta\ln\,\det F^{-1}}{\delta(F^{-1}_{bc})_{\beta\gamma}(x',x'')} = (F_{cb})_{\gamma\beta}(x',x''), \tag{4.38}$$

where we have used the formula[2]

$$\delta\left(\det F^{-1}\right) = \left(\det F^{-1}\right)\mathrm{Tr}\{F(\delta F^{-1})\}, \tag{4.39}$$

with

$$\mathrm{Tr}\{F(\delta F^{-1})\} \equiv \int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,(F_{ab})_{\alpha\beta}(x,x')\delta(F^{-1}_{ba})_{\beta\alpha}(x',x). \tag{4.40}$$

Thus the third term on the right-hand side of Eq. (4.32) is simply

$$i\hbar\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi_a(x)}(\ln\,\det F) = c^{bca}\hbar f\,\mathrm{Tr}\left\{F_{bc}(x,x)\right\}, \tag{4.41}$$

where the trace on the right-hand side is taken over the spinor indices. To evaluate the fourth term, we first note that the double-bubble diagram does not

[2] An easy way to obtain this formula is to vary the identity $\ln\{\det M\} = \mathrm{Tr}\{\ln M\}$, valid for a non-singular diagonalizable $n\times n$ matrix M. Since $\mathrm{Tr}\{\ln M\} = \sum_{i=1}^n \ln(m_i)$ with m_i's being the eigenvalues of M, we find $(\det M)^{-1}\delta(\det M) = \delta(\mathrm{Tr}\{\ln M\}) = \sum_{i=1}^n(\delta m_i)/m_i = \mathrm{Tr}\{M^{-1}\delta M\}$. By changing $M\to F^{-1}$, we obtain Eq. (4.39).

contain any $\phi_a(x)$ term, so we need to consider only the diagram with fermion propagators:

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\delta\Gamma_2}{\delta\phi_a(x)} &= -\int d^4x'\sqrt{-g'}\int d^4x''\sqrt{-g''}\,\frac{i}{2}f^2\hbar^2 c^{b'd'e'}c^{b''d''e''} \\
&\quad\times\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi_a(x)}\Big\{G_{b'b''}(x',x'')\mathrm{Tr}\{F_{d'd''}(x',x'')F_{e''e'}(x'',x')\}\Big\} \\
&= -\int d^4x'\sqrt{-g'}\int d^4x''\sqrt{-g''}\,\frac{i}{2}f^2\hbar^2 c^{b'd'e'}c^{b''d''e''}G_{b'b''}(x',x'') \\
&\quad\times\mathrm{Tr}\Big\{\frac{1}{\sqrt{-g}}\frac{\delta F_{d'd''}(x',x'')}{\delta\phi_a(x)}F_{e''e'}(x'',x') \\
&\qquad + F_{d'd''}(x',x'')\frac{1}{\sqrt{-g}}\frac{\delta F_{e''e'}(x'',x')}{\delta\phi_a(x)}\Big\}. \qquad (4.42)
\end{aligned}$$

The first term on the right-hand side of Eq. (4.42) can be evaluated as follows:

$$\begin{aligned}
&\frac{1}{\sqrt{-g}}\frac{\delta F_{d'd''}(x',x'')}{\delta\phi_a(x)}F_{e''e'}(x'',x') \\
&= \int d^4y\sqrt{-g_y}\int d^4y'\sqrt{-g'_y}\,\frac{1}{\sqrt{-g}}\frac{\delta F^{-1}_{bc}(y,y')}{\delta\phi_a(x)} \\
&\qquad\times\frac{1}{\sqrt{-g_y}}\frac{1}{\sqrt{-g'_y}}\frac{\delta F_{d'd''}(x',x'')}{\delta F^{-1}_{bc}(y,y')}F_{e''e'}(x'',x') \\
&= \int d^4y\sqrt{-g_y}\int d^4y'\sqrt{-g'_y}\left(ic^{bca}f\frac{\delta^4(y-y')}{\sqrt{-g'_y}}\frac{\delta^4(y-x)}{\sqrt{-g}}\right) \\
&\qquad\times\Big(-F_{d'b}(x',y)F_{cd''}(y',x'')\Big)F_{e''e'}(x'',x') \\
&= -ic^{bca}fF_{d'b}(x',x)F_{cd''}(x,x'')F_{e''e'}(x'',x'), \qquad (4.43)
\end{aligned}$$

where we have used the formula

$$\frac{\delta F_{ab}(x,x')}{\delta F^{-1}_{cd}(y,y')} = -\sqrt{-g_y}\sqrt{-g_{y'}}F_{ac}(x,y)F_{db}(y',x'), \qquad (4.44)$$

which can be derived by differentiating both sides of the equation

$$\int d^4x''\sqrt{-g''}\,F_{ab}(x,x'')F^{-1}_{bc}(x'',x) = \delta_{ac}\frac{\delta^4(x-x')}{\sqrt{-g}}. \qquad (4.45)$$

By interchanging $x' \leftrightarrow x''$ in the above result and taking the trace, it can be seen that the second term on the right-hand side of Eq. (4.42) is equal to the first term. Thus

$$\frac{1}{\sqrt{-g}}\frac{\delta\Gamma_2}{\delta\phi_a(x)} = -\int d^4x'\sqrt{-g'}\int d^4x''\sqrt{-g''}f^3\hbar^2 c^{b'd'e'}c^{b''d''e''}c^{bca} \times G_{b'b''}(x',x'')\mathrm{Tr}\{F_{d'b}(x',x)F_{cd''}(x,x'')F_{e''e'}(x'',x')\}. \quad (4.46)$$

Using the results in Eqs. (4.33), (4.34), (4.41) and (4.46) in Eq. (4.32), we finally obtain the dynamical equations for the mean fields:

$$\{c^{ab}(\Box+m^2)+\frac{\lambda}{6}c^{abcd}\phi_c(x)\phi_d(x)+\frac{\hbar}{2}\lambda c^{cdab}G_{cd}(x,x)\}\phi_b(x)-c^{bca}\hbar f\,\mathrm{Tr}\,\{F_{cb}(x,x)\} + \int d^4x'\sqrt{-g'}\int d^4x''\sqrt{-g''}f^3\hbar^2 c^{b'd'e'}c^{b''d''e''}c^{fga}G_{b'b''}(x',x'') \times \mathrm{Tr}\{F_{d'f}(x',x)F_{gd''}(x,x'')F_{e''e'}(x'',x')\}=0. \quad (4.47)$$

In the coincidence limit, the above equation becomes

$$\left(\Box+m^2+\frac{\lambda}{6}\phi^2(x)+\frac{\lambda\hbar}{2}G_{11}(x,x)\right)\phi(x)-\hbar f\,\mathrm{Tr}\,\{F_{11}(x,x)\}+\hbar^2 f^3\Sigma(x)=0, \quad (4.48)$$

where

$$\begin{aligned}\Sigma(y) \;=\; \int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\Big\{&G_{11}(x,x')\mathrm{Tr}\,\{F_{11}(x,y)F_{11}(y,x')F_{11}(x',x)\}\\ &-G_{21}(x,x')\mathrm{Tr}\,\{F_{21}(x,y)F_{11}(y,x')F_{12}(x',x)\}\\ &-G_{12}(x,x')\mathrm{Tr}\,\{F_{11}(x,y)F_{12}(y,x')F_{21}(x',x)\}\\ &+G_{22}(x,x')\mathrm{Tr}\,\{F_{21}(x,y)F_{12}(y,x')F_{22}(x',x)\}\Big\}. \quad (4.49)\end{aligned}$$

The dynamical equations for the propagators are obtained by setting $\delta\Gamma/\delta G_{ab}=0$:

$$\begin{aligned}0 \;=\; &\frac{\hbar}{2}\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta}{\delta G_{ab}(x,x')}\left(\int d^4x''\int d^4x'''\frac{\delta^2 S}{\delta\phi_c(x'')\delta\phi_d(x''')}G_{cd}(x'',x''')\right)\\ &-\frac{i\hbar}{2}\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta\ \ln\det G}{\delta G_{ab}(x,x')}+\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta\Gamma_2}{\delta G_{ab}(x,x')}. \quad (4.50)\end{aligned}$$

The terms on the right-hand side are evaluated as follows:

$$\begin{aligned}&\frac{\hbar}{2}\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta}{\delta G_{ab}(x,x')}\left(\int d^4x''\int d^4x'''\left(\frac{\delta^2 S}{\delta\phi_c(x'')\delta\phi_d(x''')}\right)G_{cd}(x'',x''')\right)\\ &\quad= -\frac{\hbar}{2}\left(c^{ab}(\Box+m^2)+\frac{\lambda}{2}c^{abcd}\phi_c(x)\phi_d(x)\right)\frac{\delta^4(x-x')}{\sqrt{-g'}} \quad (4.51)\end{aligned}$$

$$-\frac{i\hbar}{2}\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta}{\delta G_{ab}(x,x')}(\ln\,\det G) = -\frac{i\hbar}{2}G^{-1}_{ab}(x,x') \tag{4.52}$$

$$\begin{aligned}
&\frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta\Gamma_2}{\delta G_{ab}(x,x')}\\
&= \frac{1}{\sqrt{-g}}\frac{1}{\sqrt{-g'}}\frac{\delta}{\delta G_{ab}(x,x')}\Big(-\int d^4x''\sqrt{-g''}\,c^{cdef}\frac{\lambda\hbar^2}{4}G_{cd}(x'',x'')G_{ef}(x'',x'')\\
&\quad -\frac{i}{2}f^2\hbar^2\int d^4x''\sqrt{-g''}\int d^4x'''\sqrt{-g'''}\,c^{cde}c^{c'd'e'}\\
&\qquad\times G_{cc'}(x'',x''')\mathrm{Tr}\{F_{dd'}(x'',x''')F_{e'e}(x''',x'')\}\Big)\\
&= -c^{cdab}\frac{\lambda\hbar^2}{2}G_{cd}(x,x)\frac{\delta^4(x-x')}{\sqrt{-g'}}\\
&\quad -\frac{i}{2}f^2\hbar^2c^{ade}c^{bd'e'}\mathrm{Tr}\{F_{dd'}(x,x')F_{e'e}(x',x)\}.
\end{aligned} \tag{4.53}$$

Using the results in Eqs. (4.51), (4.52) and (4.53) in Eq. (4.50), we obtain

$$\begin{aligned}
0 = &-\frac{\hbar}{2}(c^{ab}(\Box+m^2)+\frac{\lambda}{2}c^{abcd}\phi_c(x)\phi_d(x))\frac{\delta^4(x-x')}{\sqrt{-g'}} - \frac{i\hbar}{2}G^{-1}_{ab}(x,x')\\
&-c^{cdab}\frac{\lambda\hbar^2}{2}G_{cd}(x,x)\frac{\delta(x-x')}{\sqrt{-g'}} - \frac{i}{2}f^2\hbar^2c^{ade}c^{bd'e'}\mathrm{Tr}\{F_{dd'}(x,x')F_{e'e}(x',x)\}.
\end{aligned} \tag{4.54}$$

Multiplying both sides of the above equation by $\sqrt{-g'}\,G_{be}(x',x'')$ and integrating over x', we finally obtain the dynamical equations for the propagators:

$$\begin{aligned}
-i\delta_{ae}\frac{\delta^4(x-x'')}{\sqrt{-g''}} = &\Big(c^{ab}(\Box+m^2)+\frac{\lambda}{2}c^{abcd}\phi_c(x)\phi_d(x)\\
&+c^{cdab}\lambda\hbar G_{cd}(x,x)\Big)G_{be}(x,x'')\\
&+\int d^4x'\sqrt{-g'}\,if^2\hbar c^{adf}c^{bd'f'}\mathrm{Tr}\{F_{dd'}(x,x')F_{f'f}(x',x)\}G_{be}(x',x'').
\end{aligned} \tag{4.55}$$

In the coincidence limit, the above equation for $G_{11}(x,x')$ becomes

$$\begin{aligned}
&\left(\Box+m^2+\frac{\lambda}{2}\phi^2(x)+\frac{\lambda\hbar}{2}G_{11}(x,x)\right)G_{11}(x,x')\\
&\qquad -\hbar f^2\int d^4x''\sqrt{-g''}\,\mathcal{K}(x,x'')G_{11}(x'',x') = -\frac{\delta^4(x-x')}{\sqrt{-g'}},
\end{aligned} \tag{4.56}$$

where

$$\mathcal{K}(x,x') = -i\,\mathrm{Tr}\left\{F_{11}(x,x')F_{11}(x',x) - F_{12}(x,x')F_{21}(x',x)\right\}. \tag{4.57}$$

Eqs. (4.48) and (4.56) constitute a system of integro-differential equations from which the mean fields and the propagators can be determined. Solving these equations requires the knowledge of the products of the fermion propagators; this will be discussed in the next section.

4.2 Evaluation of the Kernel

In the section, we will evaluate the kernel $\mathcal{K}(x, x')$ in Eq. (4.57) at the lowest order in perturbation, and show that the causality is satisfied by the dynamical equations for the propagators (Eq. (4.56)). For simplicity, we will work out the calculation on Minkowski space.

Since we are considering the coincidence limit, we set $\Psi_1(x) = \Psi_2(x) \equiv \Psi(x)$. Using the notations $F_{ab'} \equiv F_{ab}(x, x')$ for the fermion propagators and $\theta(x, x') \equiv \Theta(x^0 - x'^0)$ for the step function, we can write the fermion propagators in terms of the fermion correlation functions as

$$\begin{aligned} F_{11'} &= \langle\Psi(x)\overline{\Psi}(x')\rangle\theta(x, x') - \langle\overline{\Psi}(x')\Psi(x)\rangle\theta(x', x) \\ F_{22'} &= \langle\Psi(x)\overline{\Psi}(x')\rangle\theta(x', x) - \langle\overline{\Psi}(x')\Psi(x)\rangle\theta(x, x') \\ F_{12'} &= -\langle\overline{\Psi}(x')\Psi(x)\rangle \\ F_{21'} &= \langle\Psi(x)\overline{\Psi}(x')\rangle, \end{aligned} \tag{4.58}$$

which imply that

$$F_{11'} = \theta(x, x')F_{21'} + \theta(x', x)F_{12'} \tag{4.59}$$

$$F_{22'} = \theta(x', x)F_{21'} + \theta(x, x')F_{12'}. \tag{4.60}$$

Treating $F_{ab'}$ as a matrix with spinor indices as the matrix indices, for example $(F_{21'})_{\alpha\beta} \equiv \langle\Psi_\alpha(x)\overline{\Psi}_\beta(x')\rangle$, it can be verified that

$$(\mathrm{Tr}\,\{F_{11'}F_{1'1}\})^* = \mathrm{Tr}\,\{F_{22'}F_{2'2}\} \tag{4.61}$$

$$(\mathrm{Tr}\,\{F_{12'}F_{2'1}\})^* = \mathrm{Tr}\,\{F_{1'2}F_{21'}\}\,. \tag{4.62}$$

Using $\theta(x,x')\theta(x',x)=0$, we also find the relations

$$\mathrm{Tr}\left\{F_{11'}F_{1'1}\right\} = \mathrm{Tr}\left\{F_{21'}F_{1'2}\right\}\theta(x,x') + \mathrm{Tr}\left\{F_{12'}F_{2'1}\right\}\theta(x',x) \tag{4.63}$$

$$\mathrm{Tr}\left\{F_{22'}F_{2'2}\right\} = \mathrm{Tr}\left\{F_{21'}F_{1'2}\right\}\theta(x',x) + \mathrm{Tr}\left\{F_{12'}F_{21'}\right\}\theta(x,x'), \tag{4.64}$$

which tell us that the traces of the products of two fermion propagators are not all independent.

Let us now evaluate the kernel

$$\mathcal{K}(x,x') = -i\,\mathrm{Tr}\left\{F_{11'}F_{1'1} - F_{12'}F_{2'1}\right\}. \tag{4.65}$$

Using the familiar forms of the fermion propagators [15],

$$F_{11}(x,x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')}\frac{i(\not p+\mu)}{p^2-\mu^2+i\varepsilon} \tag{4.66}$$

$$F_{22}(x,x') = -\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')}\frac{i(\not p+\mu)}{p^2-\mu^2-i\varepsilon} \tag{4.67}$$

$$F_{12}(x,x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')}2\pi(\not p+\mu)\delta(p^2-\mu^2)\Theta(-p^0) \tag{4.68}$$

$$F_{21}(x,x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')}2\pi(\not p+\mu)\delta(p^2-\mu^2)\Theta(p^0), \tag{4.69}$$

we find

$$\mathrm{Tr}\left\{F_{11}(x,x')F_{11}(x',x)\right\} = -\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\; e^{-i(p-q)\cdot(x-x')}\frac{\mathrm{Tr}\{(\not p+\mu)(\not q+\mu)\}}{(p^2-\mu^2+i\varepsilon)(q^2-\mu^2+i\varepsilon)}. \tag{4.70}$$

The trace over spinor indices can be easily evaluated with the result

$$\mathrm{Tr}\{(\not p+\mu)(\not q+\mu)\} = 4(p\cdot q+\mu^2). \tag{4.71}$$

In terms of Feynman parameters [15], we can write

$$\begin{aligned}\frac{1}{(p^2-\mu^2+i\varepsilon)(q^2-\mu^2+i\varepsilon)} &= \int_0^1 d\alpha d\beta\;\delta(\alpha+\beta-1)\frac{1}{\left[\alpha(q^2-\mu^2+i\varepsilon)+\beta(p^2-\mu^2+i\varepsilon)\right]^2}\\ &= \int_0^1 d\alpha d\beta\;\delta(\alpha+\beta-1)\frac{1}{\left[p^2-\mu^2+i\varepsilon+\alpha(k^2-2p\cdot k)\right]^2}\\ &= \int_0^1 d\alpha\,\frac{1}{\left[(p-\alpha k)^2-E(\alpha;k^2+i\varepsilon)\right]^2},\end{aligned} \tag{4.72}$$

where[3]

$$E(\alpha; k^2 + i\varepsilon) \equiv \mu^2 - \alpha(1-\alpha)k^2 - i\varepsilon, \tag{4.73}$$

and we have defined $k \equiv p - q$ on the third line. Eq. (4.70) then becomes

$$\begin{aligned}
&\mathrm{Tr}\,\{F_{11}(x,x')F_{11}(x',x)\} \\
&= -4\int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\int_0^1 d\alpha\; e^{-ik\cdot(x-x')}\,\frac{(p^2 - p\cdot k + \mu^2)}{\left[(p-\alpha k)^2 - E(\alpha;k^2+i\varepsilon)\right]^2} \\
&= -4\int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\int_0^1 d\alpha\; e^{-ik\cdot(x-x')}\,\frac{(p^2 + (2\alpha-1)p\cdot k + E(\alpha;k^2+i\varepsilon))}{\left[p^2 - E(\alpha;k^2+i\varepsilon)\right]^2},
\end{aligned} \tag{4.74}$$

where we have changed $p \to p + \alpha k$ to obtain the last line. It is easy to see that the term $p\cdot k$ does not contribute to the integral, as it is an odd function of p. To evaluate the right-hand side of Eq. (4.74), we first perform an integration over p. This can be done by observing that if $\mathrm{Re}\{E(\alpha;k^2+i\varepsilon)\} = \mu^2 - \alpha(1-\alpha)k^2 > 0$, the integrand has simple poles at $p^0 = \pm\{\sqrt{|\mathbf{p}|^2 + \mathrm{Re}\{E(\alpha;k^2+i\varepsilon)\}} - i\varepsilon\}$ on the complex p^0-plane. As the integrand is proportional to $(p^0)^{-2}$ for large $|p^0|$, then we can rotate the integration contour on the complex p^0-plane counterclockwise by 90°. The procedure leads us to define a Euclidean 4-momentum variable p_E by setting $p^0 \equiv ip_E^0$ and $\mathbf{p} \equiv \mathbf{p}_E$, so that the integral over p becomes

$$\int \frac{d^4p}{(2\pi)^4}\,\frac{(p^2 + E(\alpha;k^2+i\varepsilon))}{\left[p^2 - E(\alpha;k^2+i\varepsilon)\right]^2} = -i\int \frac{d^4p_E}{(2\pi)^4}\,\frac{(p_E^2 - E(\alpha;k^2+i\varepsilon))}{\left[p_E^2 + E(\alpha;k^2+i\varepsilon)\right]^2}\,. \tag{4.75}$$

This integral clearly diverges in 4 dimensions, so we evaluate it using dimensional regularization:

$$\begin{aligned}
\int \frac{d^np}{(2\pi)^n}\,\frac{(p^2 + E(\alpha;k^2+i\varepsilon))}{\left[p^2 - E(\alpha;k^2+i\varepsilon)\right]^2} &= -i\int \frac{d^np_E}{(2\pi)^n}\,\frac{(p_E^2 - E(\alpha;k^2+i\varepsilon))}{\left[p_E^2 + E(\alpha;k^2+i\varepsilon)\right]^2} \\
&= -\frac{i}{(4\pi)^{n/2}}\left\{\frac{n}{2}\Gamma\left(1-\frac{n}{2}\right) - \Gamma\left(2-\frac{n}{2}\right)\right\} \\
&\quad \times\frac{1}{E(\alpha;k^2+i\varepsilon)^{1-n/2}}\,,
\end{aligned} \tag{4.76}$$

[3]Note that the formal definition of $E(\alpha,k^2)$ is $E(\alpha,k^2) \equiv \mu^2 - \alpha(1-\alpha)k^2$. Since $\alpha(1-\alpha) \geq 0$ for $\alpha \in [0,1]$, then it is clear we can express $E(\alpha;k^2+i\varepsilon)$ in the form in Eq. (4.73).

where we have used the formulae [15]

$$\int \frac{d^n p_E}{(2\pi)^n} \frac{1}{(p_E^2+\Delta)^k} = \frac{1}{(4\pi)^{n/2}} \frac{\Gamma(k-\frac{n}{2})}{\Gamma(k)} \left(\frac{1}{\Delta}\right)^{k-\frac{n}{2}} \tag{4.77}$$

$$\int \frac{d^n p_E}{(2\pi)^n} \frac{p_E^2}{(p_E^2+\Delta)^k} = \frac{1}{(4\pi)^{n/2}} \frac{n}{2} \frac{\Gamma(k-\frac{n}{2}-1)}{\Gamma(k)} \left(\frac{1}{\Delta}\right)^{k-\frac{n}{2}-1}. \tag{4.78}$$

To analyze Eq. (4.76) in the vicinity of $n=4$, we let $n=4-\varepsilon$. Using the formula $\Gamma(x+1)=x\Gamma(x)$ for the Gamma function, we can expand various terms in the above integral as:

$$\frac{1}{(4\pi)^{n/2}} = \frac{1}{(4\pi)^2}\left\{1+\frac{\varepsilon}{2}\ln(4\pi)+\mathcal{O}(\varepsilon^2)\right\} \tag{4.79}$$

$$\frac{1}{E(\alpha;k^2+i\varepsilon)^{1-n/2}} = E(\alpha;k^2+i\varepsilon)\left\{1-\frac{\varepsilon}{2}\ln E(\alpha;k^2+i\varepsilon)+\mathcal{O}(\varepsilon^2)\right\} \tag{4.80}$$

$$\begin{aligned}\Gamma\left(2-\frac{n}{2}\right) &= \Gamma\left(\frac{\varepsilon}{2}\right)\\ &= \frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)\end{aligned} \tag{4.81}$$

$$\begin{aligned}\Gamma\left(1-\frac{n}{2}\right) &= \Gamma\left(\frac{\varepsilon}{2}-1\right)\\ &= \frac{\Gamma(\varepsilon/2)}{(\varepsilon/2-1)}\\ &= -\left(\frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)\right)\left(1+\frac{\varepsilon}{2}+\mathcal{O}(\varepsilon^2)\right)\\ &= -\left\{\frac{2}{\varepsilon}-\gamma+1+\mathcal{O}(\varepsilon)\right\},\end{aligned} \tag{4.82}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. Using the above result, Eq. (4.76) becomes

$$\begin{aligned}\int \frac{d^n p}{(2\pi)^n} \frac{(p^2+E(\alpha;k^2+i\varepsilon))}{[p^2-E(\alpha;k^2+i\varepsilon)]^2} = -\frac{i}{(4\pi)^2}&\{3E(\alpha;k^2+i\varepsilon)\ln E(\alpha;k^2+i\varepsilon)\\ &-3E(\alpha;k^2+i\varepsilon)\left[\frac{2}{\varepsilon}-\gamma+\ln(4\pi)+\frac{1}{3}\right]\\ &+\mathcal{O}(\varepsilon)\}.\end{aligned} \tag{4.83}$$

In the limit of $\varepsilon \to 0$, the right-hand side of Eq. (4.83) contains a divergent term. To get rid of this divergence, it is convenient to use the modified minimal subtraction or $\overline{MS}$ renormalization scheme [15], in which we make a replacement

$$\left[\frac{2}{\varepsilon}-\gamma+\ln(4\pi)+\frac{1}{3}\right] \longrightarrow \ln(\Lambda^2), \tag{4.84}$$

where Λ is a mass scale; this is equivalent to imposing a series of renormalization conditions. Thus Eq. (4.83) becomes

$$\int \frac{d^n p}{(2\pi)^n} \frac{(p^2 + E(\alpha; k^2 + i\varepsilon))}{[p^2 - E(\alpha; k^2 + i\varepsilon)]^2} = -\frac{3i}{(4\pi)^2} E(\alpha; k^2 + i\varepsilon) \ln \left(\frac{E(\alpha; k^2 + i\varepsilon)}{\Lambda^2} \right). \tag{4.85}$$

Substituting the above result into Eq. (4.74), we finally obtain

$$\mathrm{Tr}\{F_{11}(x, x')F_{11}(x', x)\} = i \int \frac{d^4 k}{(2\pi)^4} e^{-ik\cdot(x-x')} A_2(k^2 + i\varepsilon), \tag{4.86}$$

where

$$A_2(k^2 + i\varepsilon) \equiv \frac{3}{4\pi^2} \int_0^1 d\alpha\, E(\alpha; k^2 + i\varepsilon) \ln \left(\frac{E(\alpha; k^2 + i\varepsilon)}{\Lambda^2} \right), \tag{4.87}$$

which is valid for the case of $\mathrm{Re}\{E(\alpha; k^2 + i\varepsilon)\} > 0$.

The result for the case of $\mathrm{Re}\{E(\alpha; k^2 + i\varepsilon)\} < 0$ can be obtained by performing an analytic continuation of the integrand of $A_2(k^2)$ in Eq. (4.87), treated as a function of $E(\alpha; k^2)$, to its values on the negative real axis of the complex $E(\alpha; k^2)$-plane. In practice, however, it is more convenient to treat $A_2(k^2)$ as a multi-valued function of complex k^0, and define the integral over k^0 as a contour integral on the complex k^0-plane. This can be done as follows. We first note that since $\alpha(1 - \alpha) \geq 0$ for $\alpha \in [0, 1]$, we can write $E(\alpha; k^2 + i\varepsilon)$ as

$$E(\alpha; k^2 + i\varepsilon) = \mu^2 - \alpha(1 - \alpha)(k^2 + i\varepsilon), \tag{4.88}$$

which enables us to think of $E(\alpha; k^2) = \mu^2 - \alpha(1 - \alpha)k^2$ as a function of complex k^0 by noting that, since $k^2 = (k^0)^2 - |\mathbf{k}|^2$, we can write

$$k^2 + i\varepsilon = \begin{cases} (k^0 + i\varepsilon)^2 - |\mathbf{k}|^2 & \text{for } k^0 > 0 \\ (k^0 - i\varepsilon)^2 - |\mathbf{k}|^2 & \text{for } k^0 < 0 \end{cases} \tag{4.89}$$

This implies that we can remove the term $i\varepsilon$ from Eqs. (4.86) and (4.87) by writing

$$\mathrm{Tr}\{F_{11}(x, x')F_{11}(x', x)\} = i \int \frac{d^4 k}{(2\pi)^4} e^{-ik\cdot(x-x')} A_2(k^2), \tag{4.90}$$

where the integral over k^0 is an integral on the complex k^0-plane, whose contour is obtained by tilting the real axis slightly in the counterclockwise direction (see Fig. 4.2). Now,

$$\begin{aligned} E(\alpha;k^2) &= \mu^2 - \alpha(1-\alpha)k^2 \\ &= -\alpha(1-\alpha)\left\{(k^0)^2 - |\mathbf{k}|^2 - \frac{1}{\alpha(1-\alpha)}\mu^2\right\} \\ &= \alpha(1-\alpha)(B-k^0)(B+k^0), \end{aligned} \tag{4.91}$$

where $B \equiv \sqrt{|\mathbf{k}|^2 + \mu^2/\alpha(1-\alpha)}$, which tells us that the function $\ln(E(\alpha;k^2))$ has two branch points at $k^0 = \pm B$ and two branch cuts from $\pm B$ to $\pm\infty$ on the real axis as shown in Fig. 4.2.

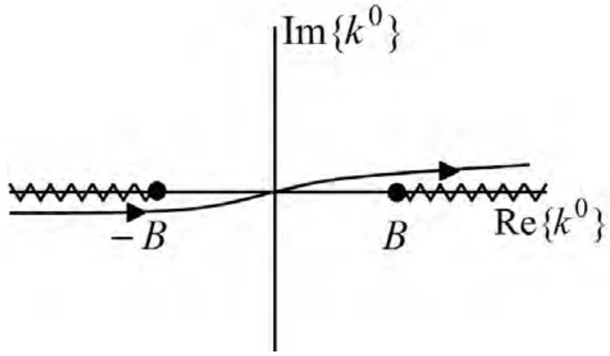

Figure 4.2: The contour of the k^0 integral in Eq. (4.90).

As the locations of the branch points depend on the value of α, one might think that we need to evaluate the k^0-integral before performing an integration over α. Such a complication, however, can be avoided as follows. Along the contour where k^0 is just above (below) the real axis for $\text{Re}\{k^0\} > 0$ ($\text{Re}\{k^0\} < 0$), by using Eq. (4.91), it is easy to verify that

$$\ln(E(\alpha;k^2)) = \ln(|E(\alpha;k^2)|) + i\pi\,\text{sgn}(k^0)\Theta(-E(\alpha;k^2)), \tag{4.92}$$

where the "sign" function is defined by

$$\text{sgn}(k^0) = \begin{cases} 1 & \text{for } k^0 > 0 \\ -1 & \text{for } k^0 < 0 \end{cases} \tag{4.93}$$

and the step function,

$$\Theta(-E(\alpha;k^2)) = \begin{cases} 0 & \text{for } E(\alpha;k^2) > 0 \text{ (or } |k^0| < \sqrt{|\mathbf{k}|^2 + \mu^2/\alpha(1-\alpha)}\ ) \\ 1 & \text{for } E(\alpha;k^2) < 0 \text{ (or } |k^0| > \sqrt{|\mathbf{k}|^2 + \mu^2/\alpha(1-\alpha)}\ ) \end{cases}, \tag{4.94}$$

tells us where the imaginary part of the function $\ln(E(\alpha;k^2))$ is non-vanishing. Thus

$$A_2(k^2) = \frac{3}{4\pi^2}\int_0^1 d\alpha\, E(\alpha;k^2)\left\{\ln\left(\frac{|E(\alpha;k^2)|}{\Lambda^2}\right) + i\pi\,\mathrm{sgn}(k^0)\Theta(-E(\alpha;k^2))\right\} \tag{4.95}$$

and the integration over k^0 is performed along the real axis. This result thus allows us to perform an integration over α before performing an integration over k^0. It is important to note that the presence of $\mathrm{sgn}(k^0)$ on the right-hand side implies the breaking of the time-reversal invariance of $\mathrm{Tr}\{F_{11}(x,x')F_{11}(x',x)\}$, which means that $\mathrm{Tr}\{F_{11}(x,x')F_{11}(x',x)\}$ is not symmetric under the interchange of x and x'.

Since the step function $\Theta(-E(\alpha;k^2))$ is non-vanishing when the conditions

$$[1-\sqrt{1-4\mu^2/k^2}] < 2\alpha < [1+\sqrt{1-4\mu^2/k^2}] \tag{4.96}$$

and $k^2 - 4\mu^2 \geq 0$ are both satisfied, then the second term of Eq. (4.95) can be easily evaluated:

$$\begin{aligned} &\frac{3i}{4\pi}\,\mathrm{sgn}(k^0)\int_0^1 d\alpha\ E(\alpha;k^2)\Theta(-E(\alpha;k^2)) \\ &\quad = \frac{3i}{4\pi}\,\mathrm{sgn}(k^0)\Theta(k^2-4\mu^2)\int_{[1-\sqrt{1-4\mu^2/k^2}]/2}^{[1+\sqrt{1-4\mu^2/k^2}]/2} d\alpha\ \left[\mu^2 - \alpha(1-\alpha)k^2\right] \\ &\quad = -\frac{i}{8\pi}\frac{k^0}{|k^0|}\,k^2\left(1-\frac{4\mu^2}{k^2}\right)^{3/2}\Theta(k^2-4\mu^2), \end{aligned} \tag{4.97}$$

where the step function $\Theta(k^2-4\mu^2)$ came from the condition $k^2-4\mu^2 \geq 0$, and we have written $\mathrm{sgn}(k^0) = k^0/|k^0|$. Thus

$$A_2(k^2) = \frac{3}{4\pi^2}\int_0^1 d\alpha\, E(\alpha;k^2)\ln\left(\frac{|E(\alpha;k^2)|}{\Lambda^2}\right) - \frac{i}{8\pi}\frac{k^0}{|k^0|}\,k^2\left(1-\frac{4\mu^2}{k^2}\right)^{3/2}\Theta(k^2-4\mu^2). \tag{4.98}$$

Substituting the above result into Eq. (4.90), we finally obtain

$$\begin{aligned} &\mathrm{Tr}\left\{F_{11}(x,x')F_{11}(x',x)\right\} \\ &= \frac{3i}{4\pi^2}\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-x')}\left\{\int_0^1 d\alpha\,E(\alpha;k^2)\ln\left(\frac{|E(\alpha;k^2)|}{\Lambda^2}\right)\right. \\ &\qquad \left. -\frac{i}{8\pi}\frac{k^0}{|k^0|}\,k^2\left(1-\frac{4\mu^2}{k^2}\right)^{3/2}\Theta(k^2-4\mu^2)\right\}. \end{aligned} \tag{4.99}$$

Using the above result, it is easy to verify that $\mathrm{Tr}\left\{F_{11}(x,x')F_{11}(x',x)\right\}$ is a pure-imaginary quantity, and so, by using

$$\mathrm{Tr}\left\{F_{22}(x,x')F_{22}(x',x)\right\} = \left(\mathrm{Tr}\left\{F_{11}(x,x')F_{11}(x',x)\right\}\right)^*,$$

we find

$$\mathrm{Tr}\left\{F_{22}(x,x')F_{22}(x',x)\right\} = -\mathrm{Tr}\left\{F_{11}(x,x')F_{11}(x',x)\right\}. \tag{4.100}$$

We next evaluate $\mathrm{Tr}\left\{F_{12}(x,x')F_{21}(x',x)\right\}$. Since $\mathrm{Tr}\left\{F_{11'}F_{1'1}+F_{22'}F_{2'2}\right\} = 0$ and $\mathrm{Tr}\left\{F_{21'}F_{1'2}\right\} = \left(\mathrm{Tr}\left\{F_{12'}F_{2'1}\right\}\right)^*$, by adding Eqs. (4.63) and (4.64) together, we find that the real part of $\mathrm{Tr}\left\{F_{12}(x,x')F_{21}(x',x)\right\}$ is zero:

$$\mathrm{Re}\,\mathrm{Tr}\left\{F_{12}(x,x')F_{21}(x',x)\right\} = 0\,. \tag{4.101}$$

Using the above result in Eq. (4.63), we find

$$\mathrm{Tr}\left\{F_{12}(x,x')F_{21}(x',x)\right\} = \mathrm{sgn}(x'^0-x^0)\,\mathrm{Tr}\left\{F_{11}(x,x')F_{11}(x',x)\right\}, \tag{4.102}$$

where

$$\begin{aligned} \mathrm{sgn}(x'^0-x^0) &\equiv \theta(x',x)-\theta(x,x') \\ &= \begin{cases} 1 & x'^0 > x^0 \\ -1 & x'^0 < x^0 \end{cases} \end{aligned} \tag{4.103}$$

Using Eqs. (4.99) and (4.102), we finally obtain the kernel:

$$\begin{aligned} \mathcal{K}(x,x') &\equiv -i\,\mathrm{Tr}\left\{F_{11}(x,x')F_{11}(x',x)-F_{12}(x,x')F_{21}(x',x)\right\} \\ &= 2\theta(x,x')\,\mathrm{Im}\,\mathrm{Tr}\left\{F_{11}(x,x')F_{11}(x',x)\right\} \\ &= \frac{3}{2\pi^2}\,\theta(x,x')\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-x')}\left\{\int_0^1 d\alpha\,E(\alpha;k^2)\ln\left(\frac{|E(\alpha;k^2)|}{\Lambda^2}\right)\right. \\ &\qquad \left. -\frac{i}{8\pi}\frac{k^0}{|k^0|}\,k^2\left(1-\frac{4\mu^2}{k^2}\right)^{3/2}\Theta(k^2-4\mu^2)\right\}. \end{aligned} \tag{4.104}$$

The fact that $\mathcal{K}(x,x')$ is proportional to $\theta(x,x')$ tells us that causality is satisfied by the dynamical equation for the propagator (Eq. (4.56)).

Let us now make some remarks about this kernel. First of all, it is not hard to see that this kernel does not mathematically make sense, as the integral defining it severely diverges, and so it is questionable if it can be used consistently in the dynamical equation for the propagator. Anyway, if we ignore this problem for a moment and look at the form of its integrand, we see that the second term is non-vanishing when $k^2 > 4\mu^2$. This condition for the non-vanishing of the second term is very interesting, since $k^0 = 2\mu$ is the threshold energy for producing two fermions of mass μ. Moreover, as this term contains a factor $k^0/|k^0|$, it breaks the time-reversal invariance of the theory. We thus expect that such a term, if its integral over d^4k makes sense at all, should describe the fermion production after the end of inflation, and therefore is responsible for the damping of the mean field of the inflaton. However, with the (unpleasant) result of this section, all we can say for now is that the mathematical structure of the nonequilibrium dynamics of a scalar field coupled to a fermion needs to be explored much more than this before we can be sure if we can rely on this theory in describing the fermion production in the universe.

CHAPTER V

CONCLUSIONS

In nonequilibrium quantum field theory, the effective action is the main quantity that we are interested in since we can determine the quantum corrections at each order in $\hbar$ and derive the dynamical equations satisfied by the mean fields and the propagators. Following the work by Hu and Ramsey [7, 8], we have obtained the effective actions and the dynamical equations for the mean fields and the propagators of a scalar field theory in curved spacetime with and without coupling with a fermion field, and a specific case of the FRW spacetime was considered where we have obtained the quantum-corrected equation of motion of the inflaton field and its propagator. An interesting point that we found is that the inflaton potential depends on its own quantum fluctuations in the form of its propagator and, at the same time, the time evolution of the propagator depends on the value of the inflaton field. It was hoped that these results should be useful in analyzing the nonequilibrium dynamics of the inflaton field and the dynamics of fermion production in the universe. Unfortunately, the dynamical equations so obtained are of the form of the integro-differential equations which cannot be solved analytically.

In the case with fermion coupling, it is expected that the integral terms in the dynamical equations should give us the damping behavior of the inflaton field whenever the condition for fermion production is satisfied. If everything goes as planed, we should be able to obtain the decay feature of the inflaton field without having to add the phenomenological damping terms to the equations of motion by hand. However, the calculation of the nonlocal kernel defined in Eq. (4.57) (which appears in the dynamical equation for the propagator) resulted in a mathematically problematic form of the kernel as the integral defining it severely diverges. Despite this problem, we have verified that, with this kernel, causality is respected by the dynamical equation for the propagator, and the time-reversal invariance is broken when the inflaton energy reaches the threshold to create a fermion pair.

The salient feature of this thesis is that the calculational result for the kernel in Eq. (4.57) is different from a similar calculation in Ref. [8], in which the authors used Cutkosky rules [15] to evaluate the other traces of products of fermion propagators from $\mathrm{Tr}\{F_{11}(x,x')F_{11}(x',x)\}$ (unfortunately, what they did is conceptually wrong). We have checked that the relationships among the traces of products of fermion propagators (derived from the definitions of fermion propagators) are not satisfied by the result obtained in Ref. [8]. Moreover, causality is not satisfied by the kernels obtained in Ref. [8]. To correct the results in Ref. [8], we firstly evaluated the term $\mathrm{Tr}\{F_{11}(x,x')F_{11}(x',x)\}$ via the traditional method of quantum field theory. In contrast with the calculation method used in Ref. [8], we have used the relationships among fermion propagators to obtain the other traces of products of fermion propagators. Instead of obtaining $\mathrm{Tr}\{F_{12}(x,x')F_{21}(x',x)\}$ in terms of the discontinuity of $\mathrm{Tr}\{F_{11}(x,x')F_{11}(x',x)\}$, we have obtained

$$\mathrm{Tr}\left\{F_{12}(x,x')F_{21}(x',x)\right\} = \mathrm{sgn}(x'^0 - x^0)\,\mathrm{Tr}\left\{F_{11}(x,x')F_{11}(x',x)\right\},$$

(see Eq. (4.102)) by using Eqs. (4.63), (4.64) and (4.100). Then, the kernel in Eq. (4.57) was found to be

$$\begin{aligned}\mathcal{K}(x,x') \;=\; & \frac{3}{2\pi^2}\,\theta(x,x')\int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-x')}\left\{\int_0^1 d\alpha\, E(\alpha;k^2)\ln\left(\frac{|E(\alpha;k^2)|}{\Lambda^2}\right)\right.\\ & \left. -\frac{i}{8\pi}\frac{k^0}{|k^0|}\,k^2\left(1-\frac{4\mu^2}{k^2}\right)^{3/2}\Theta(k^2-4\mu^2)\right\},\end{aligned}$$

(see Eq. (4.104)) which satisfies the causality requirement. We thus have obtained a new result which, we hope, will be useful in future study of nonequilibrium quantum field theory.

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VITAE

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