Chapter 2

Transport Equation

2.1 Theoretical Investigation: The Fokker-Planck Equation

Here, our attention is focused on the theory of charged particle motion in a random magnetic field in which a charged particle undergoes scattering from the magnetic irregularities, including the effects of a moving medium, the average magnetic field geometry and some energy loss processes (Ruffolo, 1995). However, it can be shown that under certain conditions this type of kinetic theoretical approach is equivalent to the Brownian motion approach (Einstein, 1906; Tajima, 1989), in which a test particle feels a fluctuating disturbance and executes a random relaxation process. The Brownian motion may be described by a Fokker-Planck equation (Chandrasekhar 1943). The details of these two approaches and

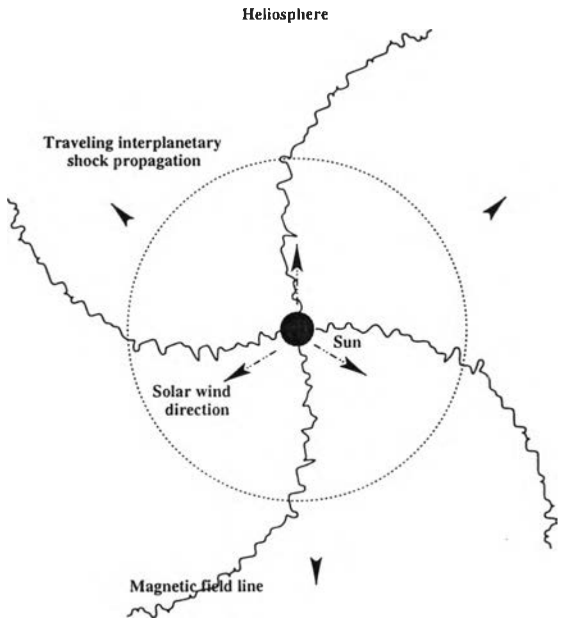

Figure 2.1: Schematic diagram of an (ideal) traveling interplanetary (spherical) shock in the heliosphere (not to scale). This shock may be driven by a coronal mass ejection (CME).

their relationship maybe found in Rosenbluth *et al.* (1957).

A standard approach to formulating the transport problem is the Fokker-Planck equation, which is solved in order to find the phase space distribution or a related quantity. Skilling (1975) provided a general transport equation (his Equation (4)) describing the behavior of charged particles under the above circumstances, valid for a wind speed much lower than the speed of light. Our situation of interest is a special case. Figure (2.1) depicts the (ideal) spherical shock geometry, possibly driven by a CME propagating outward from the Sun.

The magnetic field configuration is shown more clearly in Figure (2.2).

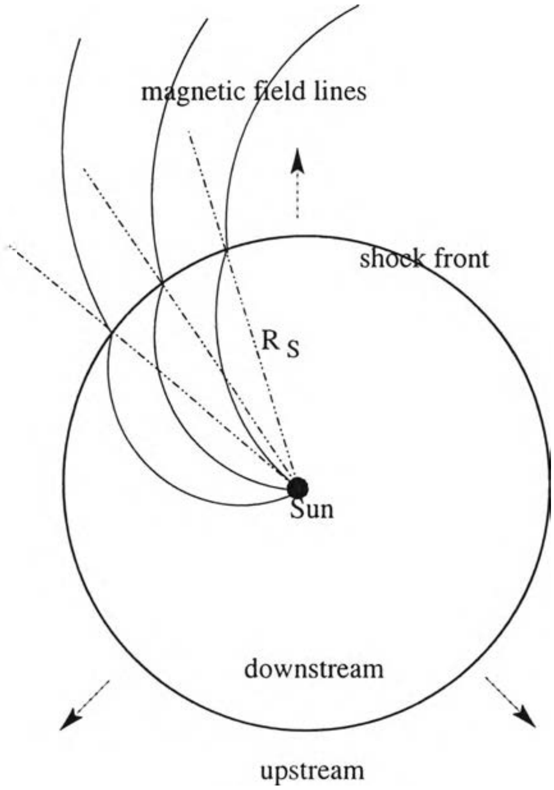

Figure 2.2: Average magnetic field lines near a spherical shock. Here the fluid inside a moving shock boundary is compressed (downstream), changing the (average) magnetic field configuration.

The equation of Skilling (1975) can be written as (Chuychai 1999):

$$\begin{aligned}\frac{\partial F}{\partial t} &= -\frac{\partial}{\partial z}\left[\mu v \ell_z + U_z - \frac{\mu^2 v^2 \vec{U}\cdot\hat{\ell}}{c^2}\ell_z\right] F \\ &\quad -\frac{\partial}{\partial \mu}\frac{1-\mu^2}{2}\left[v\,\vec{\nabla}\cdot\hat{\ell} + \mu\,\vec{\nabla}\cdot\vec{U} - 3\mu\ell_i\ell_j\frac{\partial U_j}{\partial x_i}\right. \\ &\qquad\qquad \left. +\frac{v\vec{U}}{c^2}\cdot\frac{\partial\hat{\ell}}{\partial t} - \frac{2}{v}\hat{\ell}\cdot\frac{\partial\vec{U}}{\partial t} - \frac{\mu v^2\vec{U}\cdot\hat{\ell}}{c^2}\vec{\nabla}\cdot\hat{\ell}\right] F \\ &\quad +\frac{\partial}{\partial \mu}\left[\frac{\varphi(\mu)}{2}\frac{\partial}{\partial \mu}\left(1-\frac{\mu v\,\vec{U}\cdot\hat{\ell}}{c^2}\right)F\right] \\ &\quad -\frac{\partial}{\partial p}p\left[\frac{1-3\mu^2}{2}\ell_i\ell_j\frac{\partial U_j}{\partial x_i} - \frac{1-\mu^2}{2}\vec{\nabla}\cdot\vec{U} - \frac{\mu}{v}\hat{\ell}\cdot\frac{\partial\vec{U}}{\partial t}\right]F, \qquad (2.1)\end{aligned}$$

where, $F(t,\mu,r,p) \equiv d^3N/(d\mu\, dr\, dp)$, is a distribution function related to the phase space density, $f(t,\vec{x},\vec{p})$, by $F = 2\pi r^2 p^2 f$. We use F (following Ng & Wong 1979) because we can easily design the numerical finite difference method to strictly conserve this quantity (corresponding to conservation of particles) during streaming and convection (first term of the right hand-side of Equation (2.1)). Also, μ is the pitch angle cosine of a moving particle relative to the average magnetic field, v is the particle speed, $\vec{U}$ is the fluid velocity, c is the speed of light, t is time, $\hat{\ell}$ is a unit vector along the mean magnetic field line, p is the particle momentum (magnitude), z is a spatial coordinate and $\varphi(\mu)$ is the pitch angle scattering coefficient. Actually, following Ruffolo (1995) p, μ and v are defined in the local solar wind frame while the other variables are in the solar fixed frame for convenience.

Here we modify Equation (2.1) (as a prototype to be adjusted) to suit the spherical shock case. Hence, for our shock geometry and magnetic field configuration, we have to derive the terms $\vec{U}\cdot\hat{\ell}$, $\vec{\nabla}\cdot\hat{\ell}$, $\vec{\nabla}\cdot\vec{U}$ and $\ell_i\ell_j\partial U_j/\partial x_i$. However, before going deeper in detail, the next two sections will present some basic information underlying the calculation of these quantities.

2.1.1 Solar Wind

Our simulations should be applicable, not only for a specific astrophysical situation, but also for spherical shocks in general. Since we want to study the effects of adiabatic deceleration and adiabatic focusing, it is necessary to specify some parameters in order to perform our simulations. In this case, we assume that the plasma does not flow along magnetic field lines but flows radially (or almost radially) outward from the Sun (or a star), as is typical for a solar or stellar wind. Furthermore, the solar wind (or stellar wind) speed is almost constant outside the source surface, with most of its acceleration having taken place closer to the Sun. By this point of view, we decided to use the fluid flow velocity as a vector which has a constant magnitude, radially flowing out from the center of the sphere as follows:

$$\vec{U}(r,\theta,\phi) = U_{sw}\,\hat{e}_r \tag{2.2}$$

where U_{sw} is the solar wind speed and $\hat{e}_r$ is the radial unit vector.

2.1.2 Archimedean Spiral Magnetic Field

Interplanetary magnetic field (IMF) lines have the shape of an Archimedean spiral (often called the *Parker spiral* after E. Parker, who was the first to explain the average IMF structure; Parker, 1958) when we look toward the solar equatorial plane from the solar North pole. The Archimedean spiral field in a spherical coordinate system can be written as:

$$\vec{B} = B_0 \left(\frac{R^2}{r^2}\, \hat{e}_r + \frac{R}{r}\, \hat{e}_\phi \right), \tag{2.3}$$

and the magnitude of the magnetic field is

$$|\vec{B}| = B_0 \sqrt{\left(\frac{R^2}{r^2}\right)^2 + \left(\frac{R}{r}\right)^2}, \tag{2.4}$$

where B_0 is a constant and $R = U_{sw}/(\Omega\,\sin\theta)$, is a characteristic winding radius of the Archimedean spiral.

Next, we define $\hat{\ell} = \vec{B}/|\vec{B}|$, to be a unit vector outward along the magnetic field line. Clearly, by Equations (2.3) and (2.4) we obtain

$$\hat{\ell} = \frac{\frac{R^2}{r^2}\, \hat{e}_r + \frac{R}{r}\, \hat{e}_\phi}{\sqrt{\left(\frac{R^2}{r^2}\right)^2 + \left(\frac{R}{r}\right)^2}}, \tag{2.5}$$

and (see also Figure (2.3))

$$\begin{aligned} \hat{\ell} &= \frac{R}{\sqrt{R^2 + r^2}}\, \hat{e}_r + \frac{r}{\sqrt{R^2 + r^2}}\, \hat{e}_\phi \\ &= \cos\psi\, \hat{e}_r + \sin\psi\, \hat{e}_\phi, \end{aligned} \tag{2.6}$$

where ψ is the "garden hose" angle between $\hat{\ell}$ and $\hat{r}$. Another characteristic scale length is the focusing length, $L = -B/(dB/dz)$, where z is the spatial distance along the field line. It can be shown that

$$L = \frac{r(r^2 + R^2)^{3/2}}{R(r^2 + 2R^2)}, \tag{2.7}$$

2.1.3 Specifying Terms in the Fokker-Planck Equation for a Spherical Shock

In the next step, we will provide some expressions for terms in Equation (2.1) as follows:

- **$\vec{\nabla} \cdot \hat{\ell}$ term**

  The divergence of $\hat{\ell}$ can be expressed as

$$\begin{aligned}
\vec{\nabla} \cdot \hat{\ell} &= \vec{\nabla} \cdot \frac{\vec{B}}{B} \\
&= \frac{B\vec{\nabla} \cdot \vec{B} - \vec{B} \cdot \vec{\nabla} B}{B^2} \\
&= 0 - \vec{B} \cdot \frac{\vec{\nabla} B}{B^2} \qquad \text{(since } \vec{\nabla} \cdot \vec{B} = 0\text{)} \\
&= -\frac{1}{B}\frac{\partial B}{\partial z} \\
&= \frac{1}{L}.
\end{aligned} \tag{2.8}$$

- **$\vec{\nabla} \cdot \vec{U}$ term**

  The divergence of solar wind is

$$\begin{aligned}
\vec{\nabla} \cdot \vec{U} &= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 U_r \\
&= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 U
\end{aligned}$$

$$= \frac{2U}{r}. \tag{2.9}$$

- $\vec{U} \cdot \hat{\ell}$ **term**

From Equations (2.2) and (2.6) we will directly get

$$\begin{aligned} \vec{U} \cdot \hat{\ell} &= U\, \hat{e}_r \cdot (\cos\psi\, \hat{e}_r + \sin\psi\, \hat{e}_\phi) \\ &= U\, \cos\psi \end{aligned} \tag{2.10}$$

- $\ell_i \ell_j\, \partial U_j / \partial x_i$ **term**

In the same way as expressed above we also get

$$\ell_i \ell_j \frac{\partial U_j}{\partial x_i} = \sin^2\psi\, \frac{U}{r}. \tag{2.11}$$

Substitute Equations (2.2), (2.6), (2.8), (2.9), (2.10), and (2.11) back into Equation (2.1) and using the assumption that the local solar wind speed and $\hat{\ell}$ do not depend on time ($\partial U/\partial t = 0$, $\partial\hat{\ell}/\partial t = 0$), we obtain our final transport equation.

In summary, the desired Fokker-Planck equation that describes the particle transport is:

$$\begin{aligned} \frac{\partial F}{\partial t} &= -\frac{\partial}{\partial r}\left(\mu v \cos\psi + u - \frac{\mu^2 v^2 u \cos^2\psi}{c^2}\right) F \\ &\quad -\frac{\partial}{\partial \mu}\left[\frac{v}{2L}\left(1 - \frac{\mu v u \cos\psi}{c^2}\right) + \frac{\mu u}{r}\left(1 - \frac{3}{2}\sin^2\psi\right)\right] \cdot (1-\mu^2) F \\ &\quad +\frac{\partial}{\partial \mu}\frac{\varphi}{2}\frac{\partial}{\partial \mu}\left(1 - \frac{\mu v u \cos\psi}{c^2}\right) F \\ &\quad -\frac{\partial}{\partial p}\, p u \left(\frac{1-3\mu^2}{2}\,\frac{\sin^2\psi}{r} - \frac{1-\mu^2}{r}\right) F, \end{aligned} \tag{2.12}$$

where derivation of all terms to the right is implied, and

$F(t, \mu, r, p)$ $\equiv d^3N/(dpd\mu dr)$ is the density of particles in a given magnetic flux tube,

t is the time in the fixed frame,

r is the radius,

μ is the pitch angle cosine in the fluid frame,

v is the particle speed in the fluid frame,

u is the solar wind speed in the fixed frame,

$\psi(r)$ is the "garden-hose" angle between $\vec{B}$ and $\hat{r}$,

$L(r)$ $\equiv -B/(dB/ds)$ is the focusing length, which s as the arclength along $\vec{B}$,

$\varphi(\mu)$ $= A|\mu|^{1-q}(1-\mu^2)$ is the pitch angle scattering coefficient, and

p is the particle momentum in the fluid frame.

However, from Equation (2.1) a more general form of the Fokker-Planck equation can also be written as (Ruffolo, 1995)

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial r}\left[\frac{\langle\Delta r\rangle}{\Delta t}F\right] - \frac{\partial}{\partial \mu}\left[\frac{\langle\Delta \mu\rangle}{\Delta t}F\right] + \frac{\partial}{\partial \mu}\left[\frac{\varphi(\mu)}{2}\frac{\partial}{\partial \mu}\left(1-\frac{\mu v\vec{U}\cdot\hat{\ell}}{c^2}\right)F\right]$$
$$-\frac{\partial}{\partial p}\left[\frac{\langle\Delta p\rangle}{\Delta t}F\right]. \tag{2.13}$$

Comparing this equation with Equation (2.12) term by term, we found that

$$\frac{\langle\Delta r\rangle}{\Delta t} = \mu v\cos\psi + u - \frac{\mu^2 v^2 u\cos^2\psi}{c^2}, \tag{2.14}$$

where this equation represents $\dot{r}$ and the so-called Fokker-Planck coefficient which is the description of the spatial transport velocity of charged particles we are concerned with. The first term on the right hand-side is particle streaming speed while the second term is the convection speed due to fluid flow and the last one is the relativistic correction term. Hence, this term is the overall speed of the particles we are considering in a magnetic flux tube. The next term is the effect of

$$\frac{\langle \Delta\mu \rangle}{\Delta t} = \left[\frac{v}{2L} \left(1 - \frac{\mu v u \cos\psi}{c^2} \right) + \frac{\mu u}{r} \left(1 - \frac{3}{2}\sin^2\psi \right) \right] \cdot (1 - \mu^2), \tag{2.15}$$

where this is the $\dot{\mu}$ term, which is the pitch angle changing rate due to the both effects of adiabatic focusing (the first term of the right hand-side) and the divergence of the solar wind velocity component along $\hat{\ell}$ (second term). The last term is

$$\frac{\langle \Delta p \rangle}{\Delta t} = pu \left(\frac{1 - 3\mu^2}{2} \frac{\sin^2\psi}{r} - \frac{1 - \mu^2}{r} \right). \tag{2.16}$$

This term is $\dot{p}$, which is the momentum changing rate due to the divergence of the solar wind.

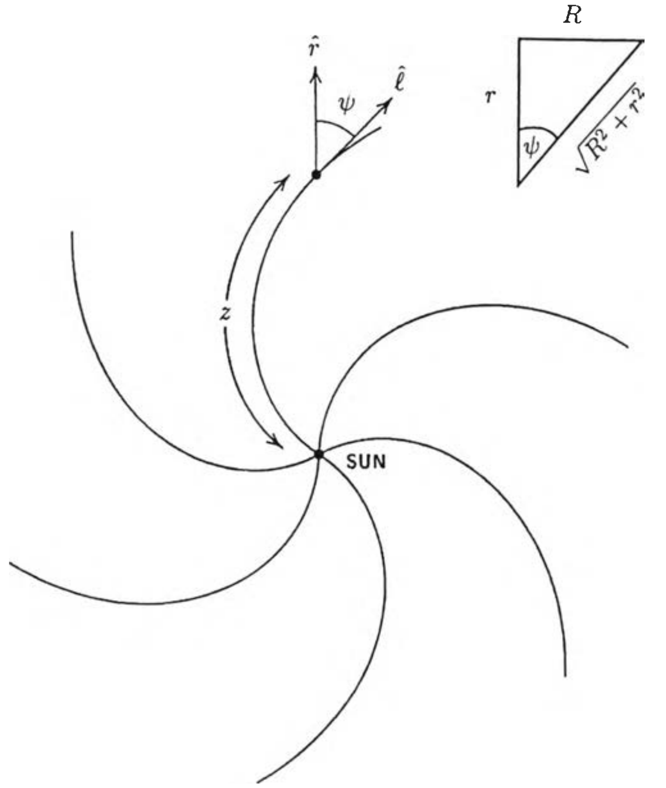

Figure 2.3: Schematic illustration of the "garden hose" angle ψ, which is the angle between the magnetic field line (along the unit vector $\hat{\ell}$) and the radial vector $\vec{r}$, where $R = U_{sw}/(\Omega \sin\theta)$ is the characteristic winding radius of the Archimedean spiral.