สูตรเชิงวิเคราะห์สำหรับโมเมนต์แบบมีเงื่อนไขของแบบจำลองเฮสตันซีอีวีไฮบริดแบบขยายกับพารามิเตอร์ที่ขึ้นกับเวลา

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ANALYTICAL FORMULA FOR CONDITIONAL MOMENTS OF EXTENDED HESTON-CEV HYBRID MODEL WITH TIME-DEPENDENT PARAMETERS

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A Thesis Submitted in Partial Fulfillment of the Requirements
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ในวิทยานิพนธ์ฉบับนี้ได้นำเสนอสูตรเชิงวิเคราะห์สำหรับโมเมนต์แบบมีเงื่อนไขของแบบจำลองเฮสตันซีอีวีแบบขยายภายใต้ไดนามิกของราคาสินทรัพย์ที่ถูกจำลองโดยกรอบการทำงานไฮบริดของกระบวนการความยืดหยุ่นของความแปรปรวนคงที่ (ซีอีวี) ซึ่งคือกระบวนการมาร์ช–โรเซนเฟลด์ สูตรนี้จะได้มาจากการแก้สมการเชิงอนุพันธ์ย่อย สูตรที่ได้นั้นจะใช้งานง่ายในเชิงปฏิบัติและอยู่ในรูปแบบทั่วไป นอกจากนี้ในวิทยานิพนธ์ฉบับนี้ได้ตรวจสอบเชิงตัวเลขเพื่อแสดงความถูกต้องของสูตรเชิงวิเคราะห์ของเราโดยเปรียบเทียบกับผลลัพธ์ที่ได้จากการจำลองแบบมอนติคาร์โล

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This thesis proposes an analytical formula for the conditional moments of the extended Heston-CEV hybrid model, which is the combination of the Heston model and the March-Rosenfeld process, also known as the constant elasticity of variance (CEV) process, to model the price dynamics of the underlying asset. The formula is derived by solving a partial differential equation (PDE) that characterizes a two-dimensional process. This formula is practical and more comprehensive than the existing results in the literature. The thesis also includes numerical validations by verifying the analytical formula against Monte Carlo simulation results to ensure its accuracy.

Department	: Mathematics and Computer Science	Student's Signature	....
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CHAPTER I

INTRODUCTION

Stochastic differential equations (SDEs) are integral equations with stochastic process term, which is a collection of random variables. Their properties such as conditional moment and variance have been well-researched. This analysis is useful for estimating parameters in various practical applications, such as pricing financial derivatives like variance swaps. For example, the risk-neutral conditional moments of returns can be used to calculate the variance swap price.

Stochastic or random processes are widely used in mathematical models to describe random phenomena, such as population growth, pollution movement, or asset price derivatives. Random processes can be organized in various ways, including the Bernoulli process, random walk, Markov process, Lévy process, Poisson process, martingale, and the most widely known Wiener process or Brownian motion [1–7].

Stochastic volatility (SV) model [8], introduced in 2015 by Bergomi, is a model that uses a randomly distributed process known as the stochastic process to evaluate option derivatives. The SV model improves upon the Black-Scholes model [9] that accounts for variable volatility. One of the main limitations of the Black-Scholes model is its assumption that volatility is constant. This is why the SV model is more popular in derivative pricing and hedging because it allows for more realistic modeling of volatility.

Heston model [10], introduced by Heston in 1993, is a well-known stochastic volatility model that describes the dynamics of an asset price. Unlike the Black-Scholes model, which assumes constant instantaneous variance, the Heston model assumes that momentary variance follows a stochastic process, specifically the Cox-Ingersoll-Ross (CIR) process. This enables the Heston model to effectively account for volatility asymmetry, the heavy-tailed nature evident in the distribution of spreading stock returns [11], and the non-negative mean-reverting volatility. The

Heston model is widely used in finance applications, such as pricing variance swaps, see [12–15], but there is evidence that the process parameters should change over time, see [16–19]. In 2011, researchers modified the Heston model by substituting the extended CIR (ECIR) instead of the fixed interest rate [20].

Constant Elasticity of Variance (CEV) process, as described in the work of Linetsky in 2010 [21], is a stochastic volatility model where the asset price's volatility is determined as a function of the power of the asset price. This power is called the elasticity factor where this power can refer to existing models such as the geometric Brownian motion, the square-root model, and the Ornstein-Uhlenbeck process, among others.

This thesis introduces an extended Heston-CEV hybrid model [22], which is a modified iteration of the Heston model that incorporates the CEV process. The formulation of this hybrid model is achieved through the solution of the partial differential equation (PDE) using the Feynman-Kac Theorem. To assess the precision of the analytical formula, we conduct a comparison between the results obtained from the analytical formula and those generated through Monte Carlo simulations.

In addition to introducing the extended Heston-CEV hybrid model, this thesis also investigates some of the essential properties of the model, including the first and second conditional moments. These properties are useful for financial applications, such as calculating the equitable strike price for a variance swap.

1.1 Literature review

SDEs find application in describing the dynamic behavior of various financial phenomena, such as derivative asset prices, interest rates, and volatility. It can also be used to study their statistical properties, including conditional moments and variance. These properties are extremely important in finance, for instance, to estimate the asset expectation and to calculate the yield of derivative contracts of underlying assets such as future contract, options contract, forward contract, and variance swaps. Some of the interesting research related to variance swaps include: in 2010, Zhu and Lian [14] introduced an analytical solution for pricing variance swaps using the two-factor stochastic volatility model proposed by Heston. In the following year, Rujivan and Zhu [23] proposed a simplified analytical formula for valuing realized variance swaps with discrete sampling, as outlined in their work.

The two articles cited above focus on developing a closed-form or analytical formula for calculating the conditional moments, which is often difficult to obtain through direct computation of the transition probability density functions (PDFs), as these PDFs are often unknown. To address this issue, the Feynman-Kac method is employed to compute the conditional moments of numerous stochastic processes. Consequently, we have further studied these aspects and found that there are many researches interested in the analytical formula of several processes. In 2015, Rujivan [24] derived the formula for the γ^{th} conditional moments of the variance process corresponding to the extended CIR process. In 2022, the analytical formula of the Heston model for calculating the moments of log prices is presented by Chumpong and Sumritnorrapong [25]. For the Heston model with other variance processes, such as the non-linear drift with a constant elasticity of variance process [26] is presented the closed-form formulas in 2022.

The March-Rosenfeld process was introducted to us by Mash and Rosenfeld [27] in 1983. This paper illustrated a new approach to validate the generalized CEV process and its parameter by using the maximum likelihood estimation and examining the pricing of default-free bonds, and then suggesting the risk premiums on default-free bonds or liquidity premiums. From here on, we will call the generalized case of the CEV process, the *"Mash-Rosenfeld process"*. From this research and the above literature review, we are inspired to try applying the Mash-Rosenfeld process, which is a nonlinear drift CEV process, as a variance process term in the Heston model. This model is referred to as the *"Heston-CEV Hybrid model"* [22]. Since the range of research about closed-form and analytical formulas for the conditional moment of various models are limited, it means that there is still no research on the extended Heston-CEV hybrid model, where all parameters of this model depend on time. We aim to derive an analytical formula for the conditional moment of this model. We have expanded the description of this model in Chapter 3.

1.2 Objectives

To derive analytical formulas and their properties for the conditional moments of the extended Heston-CEV hybrid model, and to apply these formulas to compute the conditional of moments, variance, and mixed moment, and find covariance and correlation of the model.

1.3 Structure of thesis

Chapter 2 of this thesis provides all the necessary background knowledge for the research. Chapter 3 describes the extended Heston-CEV hybrid model (3.2), including the necessary assumptions. Chapter 4 presents the main results of the analytical formula for the analytical expressions for the conditional moments of the extended Heston-CEV hybrid model and the consequences of this approach. Chapter 5 illustrates how to use the analytical formula to find essential properties of probability, include the conditional moments, volatilities, and mixed moments. Chapter 6 provides numerical validation of the analytical formulas by comparing the outcomes of conditional moments with those obtained from Monte Carlo simulations. Finally, Chapter 7 presents a conclusion of the work, discussion, and future work.

CHAPTER II

BACKGROUND KNOWLEDGE

Within this chapter, we clarify the important background knowledge and the relevant mathematical concepts related to financial processes. This chapter is divided into nine parts; the stochastic process, CIR process and ECIR process, the CEV model, the Marsh-Rosenfeld process, the Heston model, the simulations of SDE, the Feynman-Kac theorem, the Cholesky decomposition method with random variable, the Riccati equation, and the Bell polynomials, respectively.

2.1 Stochastic process

Stochastic differential equation is a type of integral equation where the coefficients are random. Therefore, SDEs are important for studying financial processes, and their concepts are presented below.

Definition 2.1. A *stochastic process* or *random process* [28], denoted as $X = \{X_t : t \in T\}$, consists of a set of random variables indexed by an index set T defined on a probability space $(\Omega, \mathcal{F}, P)$, where Ω represents the sample space, $\mathcal{F}$ is a σ-field, and P stands for the probability measure.

Definition 2.2. A stochastic process $Z = \{Z_t : t \geq 0\}$ is referred to as a *standard Brownian motion* or *Wiener process* when it meets the following conditions [28]:

(i) $Z_0 = 0$ almost surely,

(ii) Z_t is almost surely continuous in t,

(iii) The increments $Z_t - Z_s$ for any $0 \leq s < t$ are independent,

(iv) The increment $Z_t - Z_s$ follows a normal distribution with an average of 0 and a variance of $t - s$ for any $0 \leq s < t$.

Definition 2.3. A *stochastic differential equation* [29] is a specific type of integral equation where the variables are governed by stochastic processes, and it can be expressed as:

$$dX_t = \mu_t \, dt + \sigma_t \, dZ_t. \tag{2.1}$$

In this equation, μ_t represents the drift function, σ_t is the diffusion function, and Z_t is the Wiener process.

Definition 2.4. An *Itô process* [30] is a stochastic process denoted as X_t for $t \geq 0$, which can be represented as

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dZ_t, \tag{2.2}$$

where $\int_0^t \sigma_s dZ_t$ is the Itô integral. We can rewrite the previous equation as

$$dX_t = \mu_t \, dt + \sigma_t \, dZ_t. \tag{2.3}$$

Theorem 2.1 (Itô formula: simplified version [30, 31]). *Let Z_t be a Wiener process and $f(t, z)$ is a function with continuous first partial derivative w.r.t. t and second partial derivative w.r.t. z for all $t \geq 0$ and $z \in \mathbb{R}$. Then, the following holds:*

$$f(t, Z_t) = f(0, Z_0) + \int_0^t f_t(s, Z_s) \, ds + \int_0^t f_z(s, Z_s) \, dZ_s + \frac{1}{2} \int_0^t f_{zz}(s, Z_s) \, ds. \tag{2.4}$$

This formula can be expressed differentially as:

$$df(t, Z_t) = f_t(t, Z_t) \, dt + f_z(t, Z_t) \, dZ_t + \frac{1}{2} f_{zz}(t, Z_t) \, dt. \tag{2.5}$$

Theorem 2.2 (Itô formula: general version [30, 31]). *Consider a Wiener process denoted as Z_t and an Itô process X_t that satisfies the following SDE:*

$$dX_t = \mu_t \, dt + \sigma_t \, dZ_t. \tag{2.6}$$

Let $Y_t = f(t, X_t)$ be a new stochastic process, where $f(t, x)$ is a function that depends on both x and t. If $f(t, x) \in C^{1,2}([0, \infty] \times \mathbb{R})$, meaning it has continuous first partial derivative w.r.t. t and second partial derivative w.r.t. z, the process

Y_t is also an Itô process, and its dynamics can be expressed as:

$$dY_t = \left(f_t(t, X_t) + \mu_t \, f_x(t, X_t) + \frac{1}{2}\sigma_t^2 \, f_{xx}(t, X_t) \right) dt + \sigma_t \, f_x(t, X_t) \, dZ_t. \qquad (2.7)$$

2.2 CIR process and ECIR process

From the first chapter, the CIR process is introduced as a one-factor model that characterizes the dynamics of interest rates using a SDE. It serves as a tool for forecasting interest rates in the context of financial derivatives. This model, an extension of the Vasicek Interest Rate model, was originally devised in 1985 by John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross. The CIR process is precisely defined by the following SDE [16]:

$$dv_t = a\,(b - v_t)\,dt + \sigma\sqrt{v_t}\,dZ_t, \qquad (2.8)$$

where v_t signifies the instantaneous variance or the interest rate, a is the parameter of speed adjustment, b is the mean towards the long run value, σ is the volatility and Z_t is the the Wiener process, defined under a probability space $(\Omega, \mathcal{F}, P)$ along with a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$.

In fact, the monetary variables depend on time. To improve this problem, the ECIR process [16] is better suited for modeling instantaneous interest rates than the CIR process. Its form is

$$dv_t = a(t)\,(b(t) - v_t)\,dt + \sigma(t)\sqrt{v_t}\,dZ_t. \qquad (2.9)$$

In this equation, the coefficients $a(t), b(t)$ and $\sigma(t)$ are functions that depend on time.

2.3 CEV model

Constant elasticity of variance model is a popular stochastic volatility model utilized in finance, particularly for modeling equities and commodities. This model, introduced by Cox and Ross in 1976 [32], describes stochastic volatility and the associated leverage effect, which is a key feature of financial markets. The CEV model is governed by the following SDE [27]:

$$dv_t = \mu v_t \, dt + \sigma v_t^{\beta/2} \, dZ_t, \tag{2.10}$$

where v_t is the spot price, μ is the anticipated instantaneous rate of return, σ is the instantaneous volatility and β is the elasticity factor, $0 \leq \beta < 2$. When $\beta = 2$, CEV will be the geometric Brownian motion (GBM) that is used to model stock prices in the Black–Scholes model, which is the case that will be ignored. Furthermore, if $\beta = 1$, the concept is applicable to the squre-root CIR process. In practice, most options market exhibit continuous unstable volatility patterns due to implied volatility. This means that plotting the strike prices against the implied volatilities of options with the same underlying asset and expiration date typically reveals a U-shaped curve, known as a *"volatility smile"*. Empirical evidence [33] suggests that the CEV process may be more suitable for describing variance of a stock price behavior than a BS model because it can capture smiles, but the BS model is not feasible as constant volatility is an assumption of the BS model.

2.3.1 Marsh-Rosenfeld process

Marsh and Rosenfeld extended the Cox's CEV process to create the Marsh-Rosenfeld process. This was motivated by several reasons, as described in the literature [34,35]. The Marsh-Rosenfeld process is a type of CEV diffusion process as follows [27]:

$$dv_t = (A \, v_t^{-(1-\beta)} + B \, v_t) \, dt + \sigma \, v_t^{\beta/2} \, dZ_t, \tag{2.11}$$

where A and B are the coefficient of $v_t^{-(1-\beta)}$ and v_t, respectively.

Moreover, if $\beta = 1$, this process becomes the square root CIR process (2.8) with mean-reverting drift, which is introduced by Cox, Ingersoll, and Ross in 1997.

2.4 Heston model

For completing this thesis, we have reviewed the Heston model developed by the mathematician Steven Heston in 1993. This model is stochastic and developed for price options that capture the volatility behavior of an underlying asset, meaning that asset volatility is not constant, but instead changes over time in a random manner. Therefore, this model outperforms other option pricing models like the Black-Scholes model, which assumes a constant volatility. Furthermore, the Heston model is a type of volatility smile, a common graph shape pattern of implied volatility with strike price.

The primary Heston model is divided into two parts: the stochastic process governing the underlying asset price, denoted as S_t is expressed as follows [10]

$$dS_t = \mu S_t \, dt + \sqrt{v_t} S_t \, d\widetilde{Z}_t^S,$$

where the stochastic instantaneous variance v_t is given by the following CIR process

$$dv_t = a(b - v_t) \, dt + \sigma \sqrt{v_t} \, d\widetilde{Z}_t^v,$$

where μ denotes the constant interest rate or the drift of the underlying asset price returns, a is the speed of reverting to b, b represents the long-term mean of the price's variance, σ is the volatility, and $d\widetilde{Z}_t^S$ and $d\widetilde{Z}_t^v$ are interconnected Wiener processes with a correlation coefficient denoted as $\rho \in [-1, 1]$. These processes are defined within the framework of a filtered probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{0 \leq t \leq T}, Q)$ where $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is a filtration.

All parameters from the Heston model should not be constants due to time changes also affecting prices. Thus, we introduce the extended Heston model, which is an adaptation of the Heston model, substituting the constant interest rate μ with a time-varying interest rate $\mu(t)$ and replacing the CIR process with an ECIR process, respectively.

The stochastic instantaneous variance form is represented as follows [16, 17, 24]:

$$\begin{cases} dS_t = \mu(t)S_t\,dt + \sqrt{v_t}S_t\,d\widetilde{Z}_t^S, \\ dv_t = a(t)(b(t) - v_t)\,dt + \sigma(t)\sqrt{v_t}\,d\widetilde{Z}_t^v, \end{cases} \tag{2.12}$$

where $\mu(t)$, $a(t)$, $b(t)$ and $\sigma(t)$ are time dependent parameter functions.

There are three main methods for calculating the conditional moment: 1) a direct method, if the probability distribution of a random variable is ascertainable, 2) a numerical method, and 3) an analytical method. Next, we present the widely known simulation scheme, which is the prominent scheme for approximating the moment. Conversely, the Feynman-Kac theorem is used for possessing the analytical form for the moment. Both of these are shown below.

2.5 Simulation schemes for SDEs

To compare the accuracy of the analytical formula, one thing that can be checked is the numerical method to approximate the solution such as the Monte-Carlo (MC) simulation. In this thesis, we employ the Euler-Maruyama scheme, which is the easiest MC simulation for approximating the solution of an SDE. Suppose that the process S_t follows an SDE

$$dS_t = \mu(t, S_t)\,dt + \sigma(t, S_t)\,dZ_t, \tag{2.13}$$

for all $t \in [0, T]$. When we partition the time interval $[0, T]$ by N-discretized intervals, the Euler–Maruyama approximation is determined by

$$S_{t_{i+1}} = S_{t_i} + \mu(t_i, S_{t_i})\,\Delta t_i + \sigma(t_i, S_{t_i})\,\Delta Z_{t_i}, \tag{2.14}$$

where $\Delta t_i = t_{i+1} - t_i$ and $\Delta Z_i = Z_{t_{i+1}} - Z_{t_i}$ for $i \in \{0, 1, \ldots, N\}$ and ΔZ_{t_i} is estimated by $\sqrt{\Delta t_i}\,\mathcal{N}(0, 1)$, where $\mathcal{N}(0, 1)$ is the normal distribution with mean 0 and variance 1. Meanwhile, the Euler–Maruyama approximation converges to the exact solution as $N \to \infty$ and $\Delta t_i \to 0$. Nevertheless, it cannot be done in practice as it uses a lot of time to compute, and it is not possible to assign time increments close to 0 with the existing calculating devices.

To eliminate these drawbacks, there is one more challenging method, which is finding the exact solution. Then, we have to study the method that is able to lead to the solution. It is described in the next section.

2.6 Feynman-Kac theorem

In 1947, Richard Feynman and Mark Kac from Cornell constructed a formula that is able to connect PDEs and SDE. At first, this theorem described the property of physics in terms of the quantum theory "path integrals" that provide solutions to the heat equation and other diffusion equations relevant to external cooling terms. However, in finance, we utilize this theorem for finding the expectation from SDE by solving PDEs. Hence, we acheive the conditional moment that is the exact solution, aimed at mitigating the computational time expenses.

Theorem 2.3 (**N-dimensional Feyman-Kac Theorem** [36]). *Suppose that $x_{1t}, x_{2t}, \ldots, x_{Nt}$ follow*

$$\begin{aligned} dx_{1t} &= \mu_1(x_t,t)\,dt + \sum_{i=1}^{N} \sigma_{1i}(x_t,t)\,dZ_{it}, \\ dx_{2t} &= \mu_2(x_t,t)\,dt + \sum_{i=1}^{N} \sigma_{2i}(x_t,t)\,dZ_{it}, \\ &\vdots \\ dx_{Nt} &= \mu_N(x_t,t)\,dt + \sum_{i=1}^{N} \sigma_{Ni}(x_t,t)\,dZ_{it}. \end{aligned} \tag{2.15}$$

where $Z_{1t}, Z_{2t}, \ldots, Z_{Nt}$ are independent Wiener processes. Let $K \subseteq \mathbb{R}^N$, $x = (x_1, x_2, \ldots, x_N)$ and $U := U(x,t) \in C^{2,1}(K \times [0,T])$, which is dependent on both x and t. If the function satisfies the following PDE:

$$\frac{\partial U}{\partial t} + \sum_{l=1}^{N} \mu_l(x,t)\frac{\partial U}{\partial x_l} + \frac{1}{2}\sum_{l=1}^{N}\sum_{k=1}^{N} \gamma_{lk}(x,t)\frac{\partial^2 U}{\partial x_l \partial x_k} = 0, \tag{2.16}$$

where

$$\gamma_{lk}(x,t) = \sum_{i=1}^{N} \sigma_{li}(x,t)\sigma_{ki}(x,t), \tag{2.17}$$

with the terminal condition $U(x,T) = g(x)$ for all $x \in K$, which is bounded below. Then the solution $U(x,t)$ can be expressed as follows:

$$U(x,t) = \mathbb{E}[g(x_T) \mid x_t = x]. \tag{2.18}$$

For the following part, we show the relevant knowledge in the background of calculations.

2.7 Cholesky decomposition method with random variable

In our work, we employ the Cholesky decomposition method for converting the dynamical system with correlation ρ to the system with mutually independent Wiener processes. The Cholesky decomposition is as follows [37]:

$$X = L\,\eta, \tag{2.19}$$

where X is the vector of dependent random variables, L is a lower triangular matrix derived from the correlation matrix, and η is the vector consisting of independent standard normal random variables. Hence, the Cholesky decomposition of a positive definite matrix correlation C is a decomposition of the form $C = LL^\top$.

2.8 Riccati equation

The Riccati equation is essential to solve the coefficients from the ordinary differential equation belonging to the class of nonlinear differential equations of the form [38]

$$f'(x) = A(x) + B(x)\,f(x) + C(x)\,f^2(x), \tag{2.20}$$

where $A(x), C(x) \neq 0$ and $A(x), B(x), C(x)$ are continuous with respect to the variable x.

Lemma 2.1. *Observe the following Riccati equation [38]*

$$f' = X\,f^2 + Y\,f - Z, \qquad f(0,\alpha) = \alpha, \tag{2.21}$$

where X, Y, Z *and* $\alpha \in \mathbb{R}$, *with* $X \neq 0$ *and* $Y^2 + 4XZ \in \mathbb{R}$. *Suppose that* $\sqrt{\cdot}$ *is the extension of the real square root, and* $\phi = \sqrt{Y^2 + 4XZ}$.
Then, the function

$$f(t,\alpha) = -\frac{2C(e^{\phi t} - 1) - (\phi(e^{\phi t} + 1) + Y(e^{\phi t} - 1))\,u}{\phi(e^{\phi t} + 1) - Y(e^{\phi t} - 1) - 2X(e^{\phi t} - 1)\,u} \tag{2.22}$$

is unique.
Furthermore,

$$\int_0^t f(s,\alpha)\,ds = \frac{1}{X}\,\log\left(\frac{2\,\phi\,e^{\frac{\phi - B}{2}\,t}}{\phi(e^{\phi t} + 1) - Y(e^{\phi t} - 1) - 2\,X\,(e^{\phi t} - 1)\,u}\right). \tag{2.23}$$

2.9 Bell polynomials

Bell polynomials $B_n(z)$ are also called the exponential partial Bell polynomials, which are polynomials in variables $z_1, z_2, \ldots$ and is well known in the form of some particular combination sequences such as the Stirling and Bell numbers. It also appears in many applications, for example in the formula of Faà di Bruno [39].

In our work, we apply the following definition of Bell polynomials to calculate the nth conditional moment of log price.

Definition 2.5. The exponential partial Bell polynomials $\mathbf{B}_{n,j} = \mathbf{B}_{n,j}\left(z_1, z_2, \ldots, z_{n-j+1}\right)$ can be expressed by [40]

$$\mathbf{B}_{n,j}\left(z_1, z_2, \ldots, z_{n-j+1}\right) = \sum \frac{n!}{i_1! i_2! \cdots i_{n-j+1}!}\left(\frac{z_1}{1!}\right)^{i_1}\left(\frac{z_2}{2!}\right)^{i_2}\cdots\left(\frac{z_{n-j+1}}{(n-j+1)!}\right)^{i_{n-j+1}},$$

where $i_1, i_2, \cdots \geq 0$ satisfy the two conditions $\sum_{k=1}^{n-j+1} i_k = j$ and $\sum_{k=1}^{n-j+1} k i_k = n$.

This chapter presented the imperative definitions of the stochastic process including the theorems of Itô formula. In addition, we recalled the many types of variance processes, e.g., the CIR and ECIR process, the CEV model, and the Marsh-Rosenfeld process. Furthermore, we covered the Heston and extended Heston models, which are important models that we considered. Afterward, we explained how to calculate the conditional moment via simulation schemes and analytical formulas, which need to utilize the Feynman-Kac theorem for finding. For other basic knowledge, we used some essential parts of the Riccati equation, the Bell polynomial, and the Cholesky decomposition method for completing our research.

In the following chapter, we will provide a more comprehensive exposition of the extended Heston-CEV hybrid model where we merge the Marsh-Rosenfeld process into a segment of the variance process within the Heston model.

CHAPTER III

EXTENDED HESTON-CEV HYBRID MODEL

To achieve the main results, we have to define the extended Heston-CEV hybrid model, which is the model we are considering. In this chapter, we briefly overview of what this model is and what form of the SDE system can be used further to determine its conditional moment value.

3.1 Extended Heston-CEV hybrid model

From the previous chapter, we have interpreted the Heston model. The extended Heston-CEV hybrid model is also a Heston model, but all parameters depend on time, and the part of the stochastic volatility equation is changed to be a type of the CEV model. The Heston model, featuring constant parameters over time, is referred to as the extended Heston model, and its dynamics are described as follows:

$$\begin{cases} dS_t = \mu(t)\, S_t\, dt + \sqrt{v_t}\, S_t\, d\widetilde{Z}_t^S, \\ dv_t = a(t)(b(t) - v_t)\, dt + \sigma(t)\, \sqrt{v_t}\, d\widetilde{Z}_t^v, \end{cases} \tag{3.1}$$

where $\mu(t), a(t), b(t)$ and $\sigma(t)$ are time dependent parameter functions, $\widetilde{Z}_t^S$ and $\widetilde{Z}_t^v$ are interconnected Wiener processes with a correlation coefficient denoted as $\rho \in [-1, 1]$ subject to the framework of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{Q})$.

Next, we extend the Heston model (2.12) by substituting the ECIR process with a nonlinear drift process, namely, the Marsh–Rosenfeld process (2.11) with $A = a(t)\, b(t)$, $B = -a(t)$, and $\beta = \lambda$. The process S_t is described by the following dynamics [22]

$$\begin{cases} dS_t = \mu(t)\, S_t\, dt + v_t^{\frac{2-\lambda}{2}}\, S_t\, d\widetilde{Z}_t^S, \\ dv_t = a(t) \left(b(t)\, v_t^{-(1-\lambda)} - v_t \right) dt + \sigma(t)\, v_t^{\frac{\lambda}{2}}\, d\widetilde{Z}_t^v, \\ d\widetilde{Z}_t^S d\widetilde{Z}_t^v = \rho\, dt, \end{cases} \tag{3.2}$$

where $0 \le \lambda < 2$, applying Itô's lemma [41] to the first equation of (3.2) with

$x_t := \ln S_t$ and $\lambda = (2\delta - 1)/\delta$ for all $t \geq 0$. The log price process can be defined as

$$dx_t = \left(\mu(t) - \frac{1}{2}\, v_t^{\frac{1}{\delta}}\right) dt + v_t^{\frac{1}{2\delta}}\, d\widetilde{Z}_t^S,$$

where $\delta \geq 1/2$. Therefore, the system (3.2) turns into

$$\begin{cases} dx_t = \left(\mu(t) - \frac{1}{2}\, v_t^{\frac{1}{\delta}}\right) dt + v_t^{\frac{1}{2\delta}}\, d\widetilde{Z}_t^S, \\ dv_t = a(t)\left(b(t)\, v_t^{\frac{\delta-1}{\delta}} - v_t\right) dt + \sigma(t)\, v_t^{\frac{2\delta-1}{2\delta}}\, d\widetilde{Z}_t^v, \\ d\widetilde{Z}_t^S d\widetilde{Z}_t^v = \rho\, dt. \end{cases} \tag{3.3}$$

By applying the Cholesky decomposition [42] for obtaining a system with independent Wiener processes Z_t^S and Z_t^v under the measure $\mathbb{Q}$, the correlation matrix can be stated as

$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = LL^\top.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}.$$

Then,

$$LL^\top = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

From the system (3.3), it can be represented as

$$\begin{bmatrix} dx_t \\ dv_t \end{bmatrix} = \begin{bmatrix} \mu(t) - \frac{1}{2}\, v_t^{\frac{1}{\delta}} \\ a(t)\left(b(t)\, v_t^{\frac{\delta-1}{\delta}} - v_t\right) \end{bmatrix} dt + \begin{bmatrix} v_t^{\frac{1}{2\delta}} & 0 \\ 0 & \sigma(t)\, v_t^{\frac{2\delta-1}{2\delta}} \end{bmatrix} \begin{bmatrix} d\widetilde{Z}_t^S \\ d\widetilde{Z}_t^v \end{bmatrix}. \tag{3.4}$$

From the Cholesky decomposition [42],

$$X = L\,\eta.$$

Therefore,

$$\begin{bmatrix} d\widetilde{Z}_t^S \\ d\widetilde{Z}_t^v \end{bmatrix} = L \begin{bmatrix} dZ_t^S \\ dZ_t^v \end{bmatrix}.$$

From (3.4), we obtain the following matrix system

$$\begin{bmatrix} dx_t \\ dv_t \end{bmatrix} = \begin{bmatrix} \mu(t) - \frac{1}{2} v_t^{\frac{1}{\delta}} \\ a(t)\left(b(t)\, v_t^{\frac{\delta-1}{\delta}} - v_t\right) \end{bmatrix} dt + \Sigma\, L \begin{bmatrix} dZ_t^S \\ dZ_t^v \end{bmatrix},$$

where the following definitions apply to Σ and L:

$$\Sigma = \begin{bmatrix} v_t^{\frac{1}{2\delta}} & 0 \\ 0 & \sigma(t)\, v_t^{\frac{2\delta-1}{2\delta}} \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}.$$

Ultimately, the dynamical system described in equation (3.3) transforms into the following

$$\begin{cases} dx_t = \left(\mu(t) - \dfrac{1}{2} v_t^{\frac{1}{\delta}}\right) dt + v_t^{\frac{1}{2\delta}}\, dZ_t^S, \\ dv_t = a(t)\left(b(t)\, v_t^{\frac{\delta-1}{\delta}} - v_t\right) dt + \rho\sigma(t)\, v_t^{\frac{2\delta-1}{2\delta}}\, dZ_t^S + \sigma(t)\sqrt{1-\rho^2}\, v_t^{\frac{2\delta-1}{2\delta}}\, dZ_t^v. \end{cases} \tag{3.5}$$

Up to this point, we have gained this SDE system that we can use to find the conditional moment values. However, the uniqueness of the solution is still not ensured. To guarantee a strong and unique solution for the ECIR process v_t (3.5) and to prevent it from being zero almost surely, we impose the following conditions are required by Maghsoodi [17], Rogers and Williams [43], and Ekström et al. [44].

Assumption 3.1. *According to the process v_t given in (3.5), $a(t), b(t)$, and $\sigma(t)$ strictly positive and continuously differentiable with respect to time within the interval $[0, T]$ and the CEV process must satisfy $2a(t)b(t) > \sigma^2(t)$ for $\lambda = 1$ or $\sigma(t) = 0$ for $\lambda \in [0, 1) \cup (1, 2)$ [22].*

Moreover, the collection of all feasible parameter functions for the extended

Heston-CEV hybrid model (3.5) is formally characterized as follows:

$$\Big\{(\mu, a, b, \sigma, \rho) \in \big(C^1([0,T],(0,\infty)\big)^4 \times [-1,1] \mid 2a(t)b(t) > \sigma^2(t),\ \forall t \in [0,T]\Big\}. \tag{3.6}$$

A mathematical formula to calculate the conditional moments of logarithmic prices within the extended Heston-CEV hybrid model (3.5) under Assumption 3.1 in terms of log price and the parameter space (3.6) with time-dependent parameters is defined as

$$\mathbf{E}^{\mathbb{Q}}\left[x_T^n\right] := \mathbf{E}^{\mathbb{Q}}\left[x_T^n \mid \mathcal{F}_t\right] = \mathbf{E}^{\mathbb{Q}}\left[x_T^n \mid (x_T = x, v_T = v)\right], \qquad 0 \le t \le T, \tag{3.7}$$

for the degree $n \in \mathbb{N}$, our results rely on solving a partial differential equation (PDE) for a two-dimensional diffusion process [41] that produces solution solution (3.7). In general, we can solve the PDE directly for the coefficients of a polynomial expression, which gives an analytical formula.

In this chapter, we proposed the Heston model, which is part of the variance model converted to the Marsh-Rosenfeld process, which is the CEV process. Moreover, every parameter in this model is extended to be a time-dependent parameter. Thus, we are able to obtain the extended Heston-CEV hybrid model.

We have now covered the extended Heston-CEV hybrid model. In the next chapter, we will demonstrate our main results: formulas for conditional moments.

CHAPTER IV

FORMULAS FOR CONDITIONAL MOMENTS

This section derives an analytical formula for the characteristic function, denoted as U, which is an essential component for deriving the formula governing the conditional moments of the system (3.5), as outlined in Theorem 4.1, particularly in scenarios where constant parameters are observed. We also derive analytical expressions for the conditional moments of S_T, x_T, and v_T, which are provided in Corollaries 2–4, respectively. We further assess the validity of analytical formulas by comparing their results to those of MC simulations.

4.1 Formulas for conditional moments

In the following, we illustrate the conditional moment formulas for the extended Heston-CEV hybrid model and the Heston-CEV hybrid model, respectively.

4.1.1 Extended Heston-CEV hybrid model

Theorem 4.1. *Suppose* $\boldsymbol{x}(s) = (x(s), v(s))$ *follows the extended Heston-CEV hybrid model* (3.5) *on the time interval* $[t, T]$, *with* $\boldsymbol{x}(t) = (x, v)$. *The characteristic function can be written as follows [22]*

$$\mathbf{E}^{\mathbb{Q}}\left[e^{A\,x_T + B\,v_T^{\frac{1}{\delta}}} \mid (x_T = x, v_T = v)\right] =: U(x, v, \tau; A, B) = e^{\alpha(\tau;A,B)\,x + \beta(\tau;A,B)\,v^{\frac{1}{\delta}} + \eta(\tau;A,B)} \tag{4.1}$$

with $\tau = T - t$. *Then,* $\alpha(\tau; A, B) =: \alpha(\tau)$, $\beta(\tau; A, B) =: \beta(\tau)$ *and* $\eta(\tau; A, B) =:$ $\eta(\tau)$ *can be solved from the ordinary differential equations (ODEs) below*

$$\alpha'(\tau) = 0,$$

$$\beta'(\tau) = \frac{1}{2\delta^2}\,\sigma^2(t)\,\beta^2(\tau) - \frac{1}{\delta}\,\left(\rho\sigma(t)\alpha(\tau) - a(t)\right)\,\beta(\tau) + \frac{1}{2}\,(\alpha^2(\tau) - \alpha(\tau)),$$

and

$$\eta'(\tau) = \left(\frac{1}{\delta}\,a(t)b(t) + \frac{1}{2\delta^2}\,(1-\delta)\sigma^2(t)\right)\,\beta(\tau) + \mu(t)\alpha(\tau),$$

subject to the initial conditions $\alpha(0) = A$*,* $\beta(0) = B$ *and* $\eta(0) = 0$*, respectively.*

Proof. By utilizing N-dimensional Feynman-Kac theorem (2.16), we can show that U satisfies the following PDE:

$$\begin{aligned} -U_\tau &+ \left(\mu(t) - \frac{v_t^{\frac{1}{\delta}}}{2}\right) U_x + \left(a(t)\left(b(t)v_t^{\frac{\delta-1}{\delta}} - v_t\right)\right) U_v \\ &+ \tfrac{1}{2}\left(v_t^{\frac{1}{2\delta}} U_{xx} + \sigma^2(t) v_t^{\frac{2\delta-1}{\delta}} U_{vv}\right) + \rho\sigma v_t\, U_{xv} = 0, \end{aligned} \tag{4.2}$$

where $U(x, v, \tau; A, B) = \mathbf{E}^{\mathbb{Q}}\left[e^{Ax_T + Bv_T^{\frac{1}{\delta}}} \mid (x_T = x, v_T = v)\right]$. The detailed derivation of (4.2) is provided in the Appendix A.

To solve the PDE (4.2), we suppose that $U = e^{\alpha(\tau;A,B)x + \beta(\tau;A,B)v^{\frac{1}{\delta}} + \eta(\tau;A,B)}$, where α, β and η are associated functions of A, B and $\tau = T - t$. Then, all partial derivatives of U in (4.2) can be calculated as follows

$$U_\tau = \left(x\alpha' + v^{\frac{1}{\delta}}\beta' + \eta'\right)\, e^{\alpha x + \beta v^{\frac{1}{\delta}} + \eta},$$

$$U_x = \alpha\, e^{\alpha x + \beta v^{\frac{1}{\delta}} + \eta},$$

$$U_v = \tfrac{1}{\delta}\, v^{\frac{1-\delta}{\delta}}\beta\, e^{\alpha x + \beta v^{\frac{1}{\delta}} + \eta},$$

$$U_{xx} = \alpha^2\, e^{\alpha x + \beta v^{\frac{1}{\delta}} + \eta},$$

$$U_{vv} = \left(\left(\tfrac{1}{\delta}\, v^{\frac{1-\delta}{\delta}}\beta\right)^2 + \left(\tfrac{1}{\delta}\right)\left(\tfrac{1-\delta}{\delta}\right) v^{\frac{1-2\delta}{\delta}}\beta\right) e^{\alpha x + \beta v^{\frac{1}{\delta}} + \eta} \quad \text{and}$$

$$U_{xv} = \tfrac{1}{\delta}\, v^{\frac{1-\delta}{\delta}}\alpha\beta\, e^{\alpha x + \beta v^{\frac{1}{\delta}} + \eta}.$$

Replacing all partial derivatives into the PDE (4.2) and dividing it by $e^{\alpha(\tau)x+\beta(\tau)v^{\frac{1}{\delta}}+\eta(\tau)}$, it can be formulated as

$$
\begin{aligned}
-x\,\alpha' - \eta' &+ \left(\frac{1}{\delta}\,a(t)b(t) + \frac{1}{2\delta^2}\,(1-\delta)\sigma^2(t)\right)\,\beta + \mu(t)\alpha \\
&+ \left(-\beta' + \frac{1}{2\delta^2}\,\sigma^2(t)\,\beta^2 - \frac{1}{\delta}\,\left(a(t) - \rho\sigma(t)\alpha\right)\,\beta + \frac{1}{2}\,(\alpha^2-\alpha)\right) v^{\frac{1}{\delta}} = 0.
\end{aligned} \tag{4.3}
$$

To achieve the initial conditions, $\tau = 0$ or $T = t$ is determined. So, we have

$$
\begin{aligned}
e^{Ax+Bv^{\frac{1}{\delta}}} = \mathbf{E}^{\mathbb{Q}}\left[e^{Ax+Bv^{\frac{1}{\delta}}}\right] &= \mathbf{E}^{\mathbb{Q}}\left[e^{Ax_t+Bv_t^{\frac{1}{\delta}}} \mid (x_T = x, v_T = v)\right] \\
&= U(x, v, 0; A, B) = e^{\alpha(0;A,B)x+\beta(0;A,B)v^{\frac{1}{\delta}}+\eta(0;A,B)}.
\end{aligned}
$$

To compare the coefficients of the exponential term on both sides of an equation, we get $\alpha(0) = A$, $\beta(0) = B$ and $\eta(0) = 0$. From (4.3), we obtain three ODEs, that are

$$
\alpha' = 0, \tag{4.4}
$$

$$
\beta' = \frac{1}{2\delta^2}\,\sigma^2(t)\,\beta^2 - \frac{1}{\delta}\,\left(\rho\sigma(t)\alpha - a(t)\right)\,\beta + \frac{1}{2}\,(\alpha^2-\alpha) \quad \text{and} \tag{4.5}
$$

$$
\eta' = \left(\frac{1}{\delta}\,a(t)b(t) + \frac{1}{2\delta^2}\,(1-\delta)\sigma^2(t)\right)\beta + \mu(t)\alpha, \tag{4.6}
$$

subject to the initial conditions $\alpha(0) = A$, $\beta(0) = B$ and $\eta(0) = 0$, respectively.

The solutions $\alpha(\tau; A, B)$, $\beta(\tau; A, B)$ and $\eta(\tau; A, B)$ can be easily founded from the above ODEs. □

4.1.2 Heston-CEV hybrid model

Theorem 4.2. *Suppose $\boldsymbol{x}(s) = (x(s), v(s))$ follows the Heston-CEV hybrid model such that $\mu(t) = \mu, a(t) = a, b(t) = b$ and $\sigma(t) = \sigma$ with the initial value vector*

$\boldsymbol{x}(t) = (x, v)$. *The expression for the characteristic function is as follows [22]*

$$\mathbf{E}^{\mathbb{Q}}\left[e^{A\,x_T + B\,v_T^{\frac{1}{\delta}}} \mid (x_T = x, v_T = v)\right] =: U(x, v, \tau; A, B) \tag{4.7}$$
$$= e^{\alpha(\tau;A,B)\,x + \beta(\tau;A,B)\,v^{\frac{1}{\delta}} + \eta(\tau;A,B)}$$

such that the coefficients in $\tau = T - t$ *are*

$$\alpha(\tau; A, B) = A, \tag{4.8}$$
$$\beta(\tau; A, B) = -\frac{A(1-A)\left(e^{\phi\tau}-1\right) - \left(\phi\left(e^{\phi\tau}+1\right) + \left(\frac{\rho\sigma A - a}{\delta}\right)\left(e^{\phi\tau}-1\right)\right)B}{\phi\left(e^{\phi\tau}+1\right) - \left(\frac{\rho\sigma A - a}{\delta} + \left(\frac{\sigma}{\delta}\right)^2 B\right)\left(e^{\phi\tau}-1\right)} \tag{4.9}$$

and

$$\eta(\tau; A, B) = \mu A\tau \tag{4.10}$$
$$+\left(\frac{2ab\delta}{\sigma^2} - \delta + 1\right)\ln\left(\frac{2\phi\,\exp\left(\frac{\phi\delta - \rho\sigma A + a}{2\delta}\tau\right)}{\phi\left(e^{\phi\tau}+1\right) - \left(\frac{\rho\sigma A - a}{\delta} + \left(\frac{\sigma}{\delta}\right)^2 B\right)\left(e^{\phi\tau}-1\right)}\right),$$

where $\sqrt{\cdot}$ *represents the analytical continuation of the real square root with* $\phi = \sqrt{\left(\frac{\rho\sigma A - a}{\delta}\right)^2 - \left(\frac{A(1-A)\sigma^2}{\delta^2}\right)}$.

Proof. Assuming $a(t), b(t)$ and $\sigma(t)$ are constants. From (4.4)–(4.6), the system ODEs are solved, we have (4.8)–(4.10).
For finding $\alpha(\tau; A, B)$,

$$\begin{aligned} \alpha'(\tau) &= 0 \\ \int_0^{\tau} d\alpha(S) &= 0 \\ \alpha(\tau) - \alpha(0) &= 0 \\ \alpha(\tau) - A &= 0 \\ \alpha(\tau) &= A. \end{aligned}$$

Then, $\alpha(\tau; A, B) = A$.

For solving $\beta(\tau; A, B)$, we need to adapt the Riccati equations [38] to solve PDE (4.5). According to (4.5) applied with the Riccati equation (see Lemma 2.1), we derive

$$\beta(\tau; A, B) = -\frac{A(1-A)\left(e^{\phi\tau}-1\right) - \left(\phi\left(e^{\phi\tau}+1\right) + \left(\frac{\rho\sigma A - a}{\delta}\right)\left(e^{\phi\tau}-1\right)\right)B}{\phi\left(e^{\phi\tau}+1\right) - \left(\frac{\rho\sigma A-a}{\delta} + \left(\frac{\sigma}{\delta}\right)^2 B\right)\left(e^{\phi\tau}-1\right)},$$

where $\phi = \sqrt{\left(\frac{\rho\sigma A-a}{\delta}\right)^2 - \left(\frac{A(1-A)\sigma^2}{\delta^2}\right)}$.
To figure out $\eta(A, B, \tau)$, we solve the ODE (4.6).
Hence,

$$\begin{aligned}
\eta_\tau &= \left(\frac{1}{\delta}\, a(t)b(t) + \frac{1}{2\delta^2}\,(1-\delta)\sigma^2(t)\right)\beta + \mu(t)\alpha \\
\int_0^\tau d\eta(S) &= \int_0^\tau \left(\left(\frac{1}{\delta}\, a(t)b(t) + \frac{1}{2\delta^2}\,(1-\delta)\sigma^2(t)\right)\beta + \mu(t)\alpha\right) dS \\
\eta(\tau; A, B) &= \mu A\tau + \frac{a\,b}{\delta}\int_0^\tau \beta(S)dS + \frac{\sigma^2(1-\delta)}{2\,\delta^2}\int_0^\tau \beta(S)dS,
\end{aligned}$$

where, from 2.23 of Lemma 2.1,

$$\int_0^\tau \beta(S)dS = \frac{2\,\delta^2}{\sigma^2}\ln\left(\frac{2\phi\,\exp\left(\frac{\phi\delta-\rho\sigma A+a}{2\delta}\tau\right)}{\phi\left(e^{\phi\tau}+1\right) - \left(\frac{\rho\sigma A-a}{\delta} + \left(\frac{\sigma}{\delta}\right)^2 B\right)\left(e^{\phi\tau}-1\right)}\right).$$

Therefore,

$$\begin{aligned}
\eta(\tau; A, B) &= \mu A\tau \\
&\quad + \left(\frac{2ab\delta}{\sigma^2} - \delta + 1\right)\ln\left(\frac{2\phi\,\exp\left(\frac{\phi\delta-\rho\sigma A+a}{2\delta}\tau\right)}{\phi\left(e^{\phi\tau}+1\right) - \left(\frac{\rho\sigma A-a}{\delta} + \left(\frac{\sigma}{\delta}\right)^2 B\right)\left(e^{\phi\tau}-1\right)}\right).
\end{aligned}$$

□

4.2 Consequences

From the previous theorem, we suddenly obtain the following consequences when we substitute $\delta = 1$ into the extended Heston-CEV hybrid model. In addition, we obtained the formula for the conditional n-moment of the asset price S_T, the log price x_T, and the volatility v_T.

Consider $\delta = 1$, the process (3.5) can be deduced to the extended Heston model and our proposed results can be reduced to the following formulas.

Corollary 1. Setting $\delta = 1$ in Theorem 4.1, the solution of the parameters α, β, and η can be received by solving this system

$$\beta' = \frac{1}{2}\,\sigma^2(t)\,\beta^2 - \left(\rho\sigma(t)\alpha - a(t)\right)\,\beta + \frac{1}{2}\left(\alpha^2 - \alpha\right) \quad \text{and} \quad \eta' = a(t)b(t)\,\beta + \mu(t)\alpha,$$

along with the condition $\alpha(\tau; A, B) = A$, and under the initial conditions $\beta(0; A, B) = B$ and $\eta(0; A, B)$, respectively. Moreover, according Theorem 4.2, with respect to $\delta = 1$, we secure

$$\begin{aligned}
&\alpha(\tau; A, B) = A,\\
&\beta(\tau; A, B) = -\frac{A(1-A)\left(e^{\phi\tau} - 1\right) - \left(\phi\left(e^{\phi\tau} + 1\right) + (\rho\sigma A - a)\left(e^{\phi\tau} - 1\right)\right)B}{\phi\left(e^{\phi\tau} + 1\right) - (\rho\sigma A - a - \sigma^2 B)\left(e^{\phi\tau} - 1\right)} \quad \textit{and}\\
&\eta(\tau; A, B) = \mu A\tau + \frac{2ab}{\sigma^2}\ln\left(\frac{2\phi\exp\left(\frac{\phi - \rho\sigma A + a}{2}\tau\right)}{\phi\left(e^{\phi\tau} + 1\right) - (\rho\sigma A - a - \sigma^2 B)\left(e^{\phi\tau} - 1\right)}\right),
\end{aligned}$$

where $\phi = \sqrt{(\rho\sigma A - a)^2 - A(1-A)\sigma^2}$.

Proof. The proof is achieved directly by substituting $\delta = 1$ into Theorems 4.1 and 4.2, respectively. Then, we promptly gain the above corollary. □

It is important to observe that the result provided in Corollary 1 agrees with that of Theorem 1 in Chumpong [25]. Furthermore, our result can be utilized easier than Chumpong's [25] as their formula needs to be solved step by step for the recursive function coefficients, but our result simply substitutes the values of all the parameters and solve the PDEs to obtain the answer. Furthermore, the order of the conditional moment, the amount of recursive function coefficients to solve.

For the remaining three Corollaries, we derive analytical formulas for conditional moments of the asset price, log price, and volatility for the dynamics of process (3.2) using Theorem 4.1.

Corollary 2. Substituting $A = n$ and $B = 0$ into Theorem 4.1, the analytical expression for the nth conditional moment of S_T subject to $x_t = \ln S_t$ for all $t \geq 0$, where n is an integer. This formula is presented as

$$\mathbf{E}^{\mathbb{Q}}\left[S_T^n\right] = U(x, v, \tau; n, 0) = e^{\alpha(\tau:n,0)x + \beta(\tau:n,0)v^{\frac{1}{\delta}} + \eta(\tau:n,0)}. \tag{4.11}$$

Proof. From equation (4.1) in Theorem 4.1 and $x_t = \ln S_t$, we receive

$$U(x, v, \tau; A, B) = \mathbf{E}_t^{\mathbb{Q}}\left[S_T^A\, e^{B\, v_T^{\frac{1}{\delta}}}\right]. \tag{4.12}$$

Replacing $A = n$ and $B = 0$, we can have $\mathbf{E}_t^{\mathbb{Q}}[S_T^n]$. □

Corollary 3. According to Theorem 4.1, the analytical formula for the nth conditional moment of x_T for all $t \geq 0$ as follows

$$\begin{aligned}\mathbf{E}^{\mathbb{Q}}\left[x_T^n\right] &= U_A^{(n)}(x, v, \tau; 0, 0) \\ &= e^{\hat{f}} \sum_{j=1}^{n} \mathbf{B}_{n,j}\left(\hat{f}_A,\ \hat{f}_{AA},\ \ldots,\ \hat{f}_A^{(n-j+1)}\right),\end{aligned} \tag{4.13}$$

where $U_A^{(n)} := \frac{\partial^n U}{\partial A^n}$, $f(x, v, \tau; A, B) = \alpha(\tau; A, B)\, x + \beta(\tau; A, B)\, v^{\frac{1}{\delta}} + \eta(\tau; A, B)$, $\hat{f} := f(x, v, \tau; 0, 0)$ and $\mathbf{B}_{n,j}$ is the partial exponential Bell polynomials, which is defined by

$$\mathbf{B}_{n,j}\left(z_1, z_2, \ldots, z_{n-j+1}\right) = \sum \frac{n!}{i_1! i_2! \cdots i_{n-j+1}!}\left(\frac{z_1}{1!}\right)^{i_1}\left(\frac{z_2}{2!}\right)^{i_2} \cdots \left(\frac{z_{n-j+1}}{(n-j+1)!}\right)^{i_{n-j+1}},$$

where $i_1, i_2, \cdots \geq 0$ satisfy the two conditions $\sum_{k=1}^{n-j+1} i_k = j$ and $\sum_{k=1}^{n-j+1} k i_k = n$.

Proof. By using the nth derivatives with respect to A in (4.1) contained in Theorem 4.1, we get

$$U_A^{(n)}(x,v,\tau;A,B) = \mathbf{E}^{\mathbb{Q}}\left[\frac{\partial^n}{\partial A^n}\ e^{Ax_T+Bv_T^{\frac{1}{\delta}}}\right] = \mathbf{E}^{\mathbb{Q}}\left[x_T^n\ e^{Ax_T+Bv_T^{\frac{1}{\delta}}}\right]. \tag{4.14}$$

Next, we substitute $A = 0$ and $B = 0$ into (4.14). Thus, we obtain conditional moment $\mathbf{E}^{\mathbb{Q}}\left[x_T^n\right]$ as required. To obtain (4.13), Faà Bruno's formula is applied; for more details see [45]. □

Corollary 4. According to Theorem 4.1, the analytical formula for the conditional volatility of $v_T^{\frac{m}{\delta}}$ for all $t \geq 0$ can be expressed as

$$\begin{aligned}\mathbf{E}^{\mathbb{Q}}\left[v_T^{\frac{m}{\delta}}\right] &= U_B^{(m)}(x,v,\tau;0,0)\\ &= e^{\hat{f}}\sum_{j=1}^{m}\mathbf{B}_{m,j}\left(\hat{f}_B,\ \hat{f}_{BB},\ \ldots,\ \hat{f}_B^{(m-j+1)}\right),\end{aligned} \tag{4.15}$$

where $U_B^{(m)} := \frac{\partial^m U}{\partial B^m}$ and $\mathbf{B}_{m,j}$ is the partial exponential Bell polynomials.

Proof. This proof is likewise the previous Corollary. □

We have presented Theorems and Corollaries. After that, we will investigate our results by numerical validation. Additionally, we will provide examples of conditional moments of log asset price along with the conditional volatilities in the next chapter.

CHAPTER V

PROPERTIES OF PROBABILITY

This chapter demonstrates examples of utilizing analytical formulas to find various types of conditional moments. Moreover, we will show applications of our formulas with properties of probability, including the fractional conditional volatilities, variance moment, mixed moment, covariance, and correlation of the Heston-CEV hybrid model.

Example 5.1 (The conditional moments)**.** By applying Corollary 3, we obtain the following closed-form formula for the first conditional moment as follows:

$$\mathbf{E}^{\mathbb{Q}}\left[x_T\right] = U_A(x, v, \tau; 0, 0) = e^{\hat{f}}\,\hat{f}_A, \tag{5.1}$$

where

$$\begin{aligned}\hat{f}_A = x &- \frac{1}{2a}\,\delta(1 - e^{\frac{-a\tau}{\delta}})\,v^{\frac{1}{\delta}} \\ &+ \frac{1}{4\delta a^2}\,\left((\delta-1)\sigma^2 - 2\delta ab\right)\left((e^{\frac{-a\tau}{\delta}} - 1)\delta + a + 4\delta a^2\mu\right).\end{aligned}$$

Meanwhile, the closed-form formula for the second conditional moment is as follows:

$$\begin{aligned}\mathbf{E}^{\mathbb{Q}}\left[x_T^2\right] = U_A^{(2)}(x, v, \tau; 0, 0) &= e^{\hat{f}}\,\hat{f}_{AA} + \hat{f}_A\,e^{\hat{f}}\,\hat{f}_A \\ &= e^{\hat{f}}\,\left(\hat{f}_{AA} + \left(\hat{f}_A\right)^2\right),\end{aligned} \tag{5.2}$$

where

$$
\begin{aligned}
\hat{f}_{AA} = \frac{e^{-\frac{2a\tau}{\delta}}}{16\delta a^4} & \Big(-\delta\sigma^2 \left(-2\delta ba + (\delta-1)\sigma^2 + 4\delta a v^{1/\delta}\right) \\
& + \ 4e^{\frac{a\tau}{\delta}} \Big(-2\delta a^2 v^{1/\delta}(2\delta(a-\rho\sigma) + \sigma\tau(\sigma - 2a\rho)) \\
& + \left(2\delta ba - \delta\sigma^2 + \sigma^2\right)\left(\delta\left(2a^2 - 4a\rho\sigma + \sigma^2\right) + a\sigma\tau(\sigma - 2a\rho)\right)\Big) \\
& + \ e^{\frac{2a\tau}{\delta}} \Big(4\delta^2 a v^{1/\delta}\left(4a^2 - 4a\rho\sigma + \sigma^2\right) \\
& - \ \left(2\delta ba - \delta\sigma^2 + \sigma^2\right)\left(\delta\left(8a^2 - 16a\rho\sigma + 5\sigma^2\right) - 2a\tau\left(4a^2 - 4a\rho\sigma + \sigma^2\right)\right)\Big)\Big).
\end{aligned}
$$

Example 5.2 (The fractional conditional volatilities)**.** By applying Corollary 4, the closed-formula of fractional conditional volatility $v_T^{\frac{m}{\delta}}$ of the Heston-CEV hybrid model with $m = 1$ and 2 is given by

$$
\mathbf{E}^{\mathbb{Q}}\left[v_T^{\frac{1}{\delta}}\right] = U_B(x, v, \tau; 0, 0) = e^{\hat{f}}\,\hat{f}_B, \tag{5.3}
$$

$$
\mathbf{E}^{\mathbb{Q}}\left[v_T^{\frac{2}{\delta}}\right] = U_B^{(2)}(x, v, \tau; 0, 0) = e^{\hat{f}}\,\left(\hat{f}_{BB} + \left(\hat{f}_B\right)^2\right), \tag{5.4}
$$

where

$$
\begin{aligned}
\hat{f}_B &= \frac{e^{-\frac{a\tau}{\delta}}}{2\delta a}\left(\left(e^{\frac{a\tau}{\delta}} - 1\right)\left((\delta-1)\sigma^2 - 2\delta ab\right) + 2\delta a v^{1/\delta}\right), \\
\hat{f}_{BB} &= -\frac{\sigma^2 e^{-\frac{2a\tau}{\delta}}}{4\delta^2 a^2}\left(e^{\frac{a\tau}{\delta}} - 1\right)\left(\left(e^{\frac{a\tau}{\delta}} - 1\right)\left((\delta-1)\sigma^2 - 2\delta ab\right) + 4\delta a v^{1/\delta}\right).
\end{aligned}
$$

In the next example, we will present the conditional variance, mixed moments, covariance, and correlation, respectively, as consequences of Example 5.1.

Example 5.3 (The conditional variance moment)**.** To find the conditional variance of the Heston-CEV hybrid model, we can apply Equations (5.1) and (5.2) from Example 5.1. The conditional variance can be represented as follows:

$$
\begin{aligned}
\mathbf{Var}\left[x_T \mid (x_T = x, v_T = v)\right] &= \mathbf{E}^{\mathbb{Q}}\left[(x_T - \mathbf{E}^{\mathbb{Q}}\left[x_T\right])^2 \mid (x_T = x, v_T = v)\right] \\
&= \mathbf{E}^{\mathbb{Q}}\left[x_T^2\right] - \left(\mathbf{E}^{\mathbb{Q}}\left[x_T\right]\right)^2 \\
&= e^{\hat{f}}\left(\hat{f}_{AA} + \left(\hat{f}_A\right)^2\right) - \left(e^{\hat{f}}\,\hat{f}_A\right)^2 \\
&= e^{\hat{f}}\left(\hat{f}_{AA} + (1 - e^{\hat{f}})\left(\hat{f}_A\right)^2\right).
\end{aligned}
$$

Example 5.4 (The conditional mixed moment)**.** From Corollaries 3 and 4, the conditional mixed moment of the Heston-CEV hybrid model is given by

$$
\begin{aligned}
\mathbf{E}^{\mathbb{Q}}\left[x_T v_T^{\frac{1}{\delta}}\right] = U_{AB}^{(1,1)}(x, v, \tau; 0, 0) &= e^{\hat{f}}\,\hat{f}_{AB} + \hat{f}_A\, e^{\hat{f}}\,\hat{f}_B \\
&= e^{\hat{f}}\left(\hat{f}_{AB} + \hat{f}_A \hat{f}_B\right),
\end{aligned} \tag{5.5}
$$

where

$$
\begin{aligned}
\hat{f}_{AB} = &\frac{v^{\frac{1}{\delta}}}{2\delta a^2}\left(\sigma e^{-\frac{a\tau}{\delta}}\left(\delta\sigma\left(e^{\frac{a\tau}{\delta}} - 1\right) + a\tau(2a\rho - \sigma)\right)\right) \\
&- \frac{1}{4\delta^2 a^3}\,\sigma e^{-\frac{a\tau}{\delta}}\left(2\delta ab - (\delta - 1)\sigma^2\right) \\
&\left(\delta(2a\rho - \sigma)\sinh\left(\frac{a\tau}{\delta}\right) + 2\delta a\rho\cosh\left(\frac{a\tau}{\delta}\right) - 2\delta a\rho + a\tau(\sigma - 2a\rho)\right),
\end{aligned}
$$

Through Examples 5.1, 5.2 and 5.4, we can apply these examples for calculating the K_{var} value of the pricing variance swap contract, which is a derivative contract that allows investors to buy or sell the anticipated future variance of the log-returns of an underlying asset.

Example 5.5 (The conditional covariance)**.** By applying (5.1) and (5.2) from Example 5.1 to find the conditional covariance of x and v with $\delta = 1$, it can be given by

$$
\begin{aligned}
\mathbf{Cov}\left[x_T, v_T \mid (x_T = x, v_T = v)\right] &= \mathbf{E}^{\mathbb{Q}}\left[(x_T - \mathbf{E}^{\mathbb{Q}}\left[x_T\right])(v_T - \mathbf{E}^{\mathbb{Q}}\left[v_T\right]) \mid (x_T = x, v_T = v)\right] \\
&= \mathbf{E}^{\mathbb{Q}}\left[x_T\, v_T\right] - \mathbf{E}^{\mathbb{Q}}\left[x_T\right]\, \mathbf{E}^{\mathbb{Q}}\left[v_T\right] \\
&= e^{\hat{f}}\, \left(\hat{f}_{AB} + \hat{f}_A \hat{f}_B\right) - \left(e^{\hat{f}}\, \hat{f}_A\right)\, \left(e^{\hat{f}}\, \hat{f}_B\right) \\
&= e^{\hat{f}}\, \hat{f}_{AB} + \left(e^{\hat{f}} - (e^{\hat{f}})^2\right)\, \hat{f}_A \hat{f}_B \\
&= e^{\hat{f}}\, \left(\hat{f}_{AB} + (1 - e^{\hat{f}})\, \hat{f}_A \hat{f}_B\right).
\end{aligned}
$$

Example 5.6 (The conditional correlation)**.** By applying (5.1) and (5.2) from Example 5.1 to find the conditional correlation of x and v with $\delta = 1$, it can be given by

$$
\begin{aligned}
\mathbf{Corr}\left[x_T \mid (x_T = x, v_T = v)\right] &= \frac{\mathbf{Cov}\left[x_T, v_T \mid (x_T = x, v_T = v)\right]}{\mathbf{Var}\left[x_T \mid (x_T = x, v_T = v)\right]^{1/2}\, \mathbf{Var}\left[v_T \mid (x_T = x, v_T = v)\right]^{1/2}} \\
&= \frac{e^{\hat{f}}\, \left(\hat{f}_{AB} + (1 - e^{\hat{f}})\, \hat{f}_A \hat{f}_B\right)}{\left(e^{\hat{f}}\, \left(\hat{f}_{AA} + (1 - e^{\hat{f}})\left(\hat{f}_A\right)^2\right)\right)^{1/2}\, \left(e^{\hat{f}}\, \left(\hat{f}_{BB} + (1 - e^{\hat{f}})\left(\hat{f}_B\right)^2\right)\right)^{1/2}} \\
&= \frac{\hat{f}_{AB} + (1 - e^{\hat{f}})\, \hat{f}_A \hat{f}_B}{\left(\hat{f}_{AA} + (1 - e^{\hat{f}})\left(\hat{f}_A\right)^2\right)^{1/2}\, \left(\hat{f}_{BB} + (1 - e^{\hat{f}})\left(\hat{f}_B\right)^2\right)^{1/2}}.
\end{aligned}
$$

The above examples mean that our formula can be applied statistically and effectively. In the following chapter, we will examine our analytical formulas by using a numerical method to compare their results from Monte Carlo simulations with the results from our formula.

CHAPTER VI

NUMERICAL VALIDATION EXPERIMENT

In this chapter, we numerically validate the accuracy of our closed-form formula for the Heston-CEV hybrid model (3.2) from example 5.1, 5.2 and 5.4 by applying the Euler–Maruyama method.

6.1 Accuracy of analytical formula

Suppose that $\hat{x}_t$ and $\hat{v}_t$ be time division estimation of x_t and v_t over the interval $[0, T]$ in N steps. Therefore, the Euler–Maruyama algorithm applied to equation (3.2) can be approximated as follows

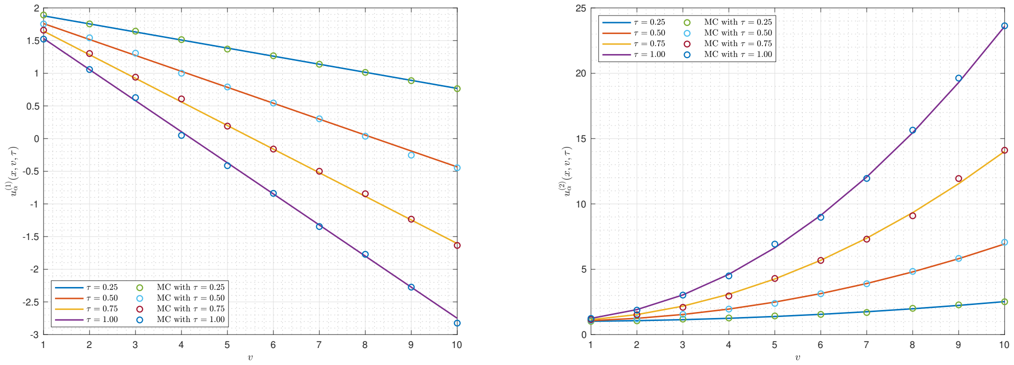
$$\begin{cases} \hat{x}_{t_i} = \hat{x}_{t_{i-1}} + \left(\mu(t_{i-1}) - \frac{1}{2}\hat{v}_{t_{i-1}}^{\frac{1}{\delta}}\right) dt + \hat{v}_{t_{i-1}}^{\frac{1}{2\delta}} \sqrt{\Delta t}\, Z_{i+1}^{(1)}, \\ \hat{v}_{t_i} = \hat{v}_{t_{i-1}} + a(t_{i-1}) \left(b(t_{i-1})\, \hat{v}_{t_{i-1}}^{\frac{\delta-1}{\delta}} - \hat{v}_{t_{i-1}}\right) dt + \rho\sigma(t_{i-1})\, \hat{v}_{t_{i-1}}^{\frac{2\delta-1}{2\delta}}\, Z_{i+1}^{(1)} \\ \qquad + \sigma(t_{i-1})\sqrt{1-\rho^2}\, \hat{v}_{t_{i-1}}^{\frac{2\delta-1}{2\delta}} \sqrt{\Delta t}\, Z_{i+1}^{(2)}, \end{cases} \tag{6.1}$$

where the initial value $\hat{x}_{t_0} = x_{t_0}$ and $\hat{v}_{t_0} = v_{t_0}$ with $\Delta t = t_i - t_{i-1}$ is a constant time step size, and $Z_i^{(1)}$ and $Z_i^{(2)}$ are normally distributed and independent.

In this thesis, we implement the numerical simulations via MATLAB R2021b, which runs on a computer that is configurable to 1.1GHz quad-core Intel Core i5-LPDDR4X, Turbo Boost up to 3.5GHz with 6MB L3 cache, speed of 3733MHz, 16.0GB RAM, 512GB SSD.

6.1.1 The conditional moments

We set the parameters corresponding to assumption 3.1, namely $\mu = 0.01$, $a = 0.1$, $b = 0.1$, $\rho = 0.01$, $\sigma = 0.001$, and $\delta = 1$ are set for the comparison of results between the formulas (5.1)–(5.2) from example 5.1 and the MC simulations with 10,000 sample paths and consisting of 10,000 steps shown in Figure 6.1. These simulations consider results for different time $\tau = 0.25, 0.5, 0.75, 1$ subject to initial values $x = 1$ and v ranges from 1 to 10.



(a) The first conditional moment **(b)** The second conditional moment

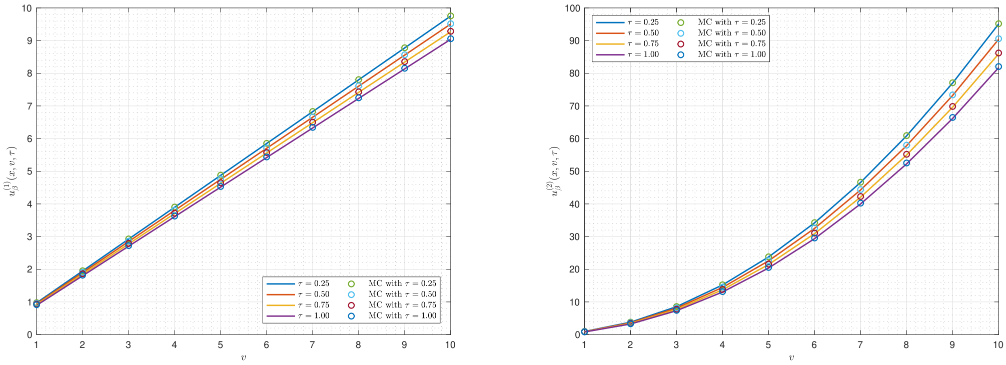
Figure 6.1: Validation testings of the first and second conditional moments at different times $\tau = 0.25, 0.5, 0.75, 1$.

According to Figure 6.1, the colored circles illustrate the MC simulation predictions that completely assort with the colored lines, which are the results from the closed-form expressions for the first and second conditional moments (5.1)–(5.2) from Example 5.1.

Despite perfect compatibility, the significant disadvantage of the MC simulation is the time-consuming calculations required for estimation.

6.1.2 The fractional conditional volatilities

By validating the formulas (5.3) and (5.4) from Example 5.2, the comparisons of the accuracy of the first and second conditional volatility with the MC simulations are represented as the figure 6.2



(a) The first conditional volatility

(b) The second conditional volatility

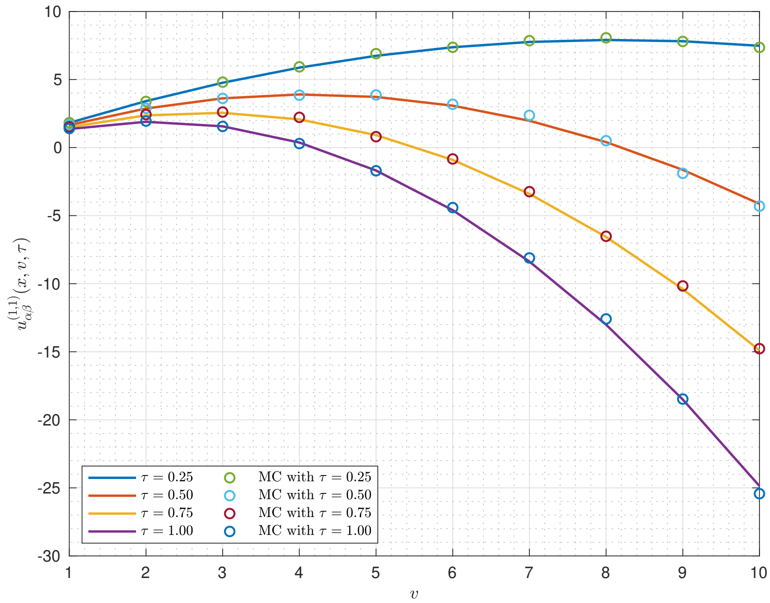
Figure 6.2: Validation testings of the first and second conditional volatilities at different times $\tau = 0.25, 0.5, 0.75, 1$.

From the considered line graph, it appears that there is an absolute pairing between the outcomes of the MC simulations and our approach. Therefore, it is apparent that our closed-form formulas are evidently correct.

From Figures 6.1 and 6.2, we can consider the first order from both figures to be linear graphs. Additionally, the second order is a parabola due to the polynomial. Furthermore, it can be seen that utilizing more value τ, values of the conditional volatilities will decrease accordingly.

6.1.3 The conditional mixed moment

Moving on to the next point, we examined the example of the conditional mix moment (5.5). The figure 6.3 shows that our results are in excellent agreement with the MC simulations.



(a) The conditional mix moment

Figure 6.3: Validation testings of the conditional mix moment at different times $\tau = 0.25, 0.5, 0.75, 1$.

In conclusion, the general overview here is that our results are correct due to a perfect match with the MC approach. Moreover, the results from our approach are better than the results from the MC simulations due to our closed-form formulas which can find the solutions with the continuous initial v value. Meanwhile, the MC simulations can only find the solutions of some initial v value that we set because it can calculate the results individually from the given initial values.

After this, we will demonstrate any other reasons why our results are more effective than the MC simulations in the nearby subsection.

6.2 Percentage relative errors and computational times

Up to this point, we demonstrated an analytical approach for earning closed-form formulas for the first and second conditional moments of the stochastic volatility model, based on the Heston-CEV hybrid model. We also validated the accuracy of our approach by comparing it to the Monte Carlo simulation approach using the Euler–Maruyama method. Our results demonstrate the superior performance of our approach compared to the MC simulation approach.

In this subsection, we will show the computational times evidence that can ensure our approach is indeed better than MC. Also, we will illustrate the percentage of relative errors that can occur if the MC simulations are used.

6.2.1 Percentage relative errors

From the previous section, we still employ $\mu = 0.01$, $a = 0.1$, $b = 0.1$, $\rho = 0.01$, $\sigma = 0.001$, $\delta = 1$, for different time $\tau = 0.25, 0.5, 0.75, 1$ subject to initial values $x = 1$ and v ranges from 1 to 10 in this subsection. However, we consider this experiment on various cases of sample paths including 5,000, 10,000, 20,000, and 100,000 paths.

Next, the percentage relative error is defined as

$$\varepsilon(v) = \left| \frac{\mathbf{E}^{\mathbb{Q}}\left[x_T\right](v) - \mathbf{E}^{\mathbb{Q}}\left[x_T\right]_{MC}(N_p, v)}{\mathbf{E}^{\mathbb{Q}}\left[x_T\right](v)} \right| \times 100\,\%,$$

where $\mathbf{E}^{\mathbb{Q}}\left[x_T\right](v)$ satisfies our exact solution of the first conditional moment with the initial v. On the contrary, $\mathbf{E}^{\mathbb{Q}}\left[x_T\right]_{MC}(N_p, v)$ is the approximate of the first conditional moment obtained by the MC simulations with the initial v simulating the N_p sample paths.

As displayed in the following Table 6.1, we illustrate the percentage relative errors of using the MC simulations for approximating the first conditional moment when compared with using our formulas for different time $\tau = 0.25, 0.5, 0.75, 1$ subject to initial values $x = 1$ and v ranges from 1 to 10.

N_p	$v=1$	$v=2$	$v=3$	$v=4$	$v=5$	$v=6$	$v=7$	$v=8$	$v=9$	$v=10$
	$\tau=0.25s$									
5,000	0.6635	0.0080	0.7498	0.3916	1.0652	0.4659	0.0328	0.1660	0.5164	0.4941
10,000	0.2071	0.8059	0.1426	1.0488	0.9293	1.2369	0.2776	0.6040	1.6178	0.3156
20,000	0.2776	0.4031	0.7444	0.2271	0.2583	0.5766	0.4736	2.7791	0.7119	0.7341
100,000	0.1251	0.1111	0.0338	0.1113	0.0610	0.3084	0.4065	0.4690	0.8621	0.4352
	$\tau=0.5s$									
5,000	0.3785	1.6991	2.7049	2.7511	0.6023	0.5620	1.7882	32.4701	33.4929	2.8995
10,000	0.0081	0.0971	0.8507	0.5793	1.5054	1.6699	1.3841	51.6985	2.2071	8.5060
20,000	0.0783	0.6705	0.4510	1.0918	0.2882	0.5037	3.7353	51.8784	14.1067	1.8335
100,000	0.0158	0.1980	0.1947	0.4964	0.0235	0.4840	0.5708	9.9372	1.6691	0.9304
	$\tau=0.75s$									
5,000	0.8958	1.4286	1.8974	8.2104	4.4402	0.7078	4.3391	4.4764	0.9383	1.7615
10,000	0.4281	0.2148	0.8994	6.2377	2.7000	5.5944	1.6396	5.0996	0.7592	1.4331
20,000	0.7490	1.1659	0.2189	1.0829	7.1483	4.8870	7.3032	3.0951	2.3945	0.4897
100,000	0.0705	0.1117	0.0715	0.3773	0.8438	5.0778	2.2556	0.5440	0.4444	0.0613
	$\tau=1.0s$									
5,000	0.6248	0.1277	8.0784	53.3146	11.5751	1.0530	1.7747	1.5392	0.0375	2.6886
10,000	0.6429	1.6410	5.2908	22.3894	6.2342	3.2924	0.9442	0.6247	0.0179	0.0888
20,000	0.2727	0.9152	4.1625	2.3975	0.7409	2.3657	0.1811	0.0512	0.7891	0.3618
100,000	0.0529	0.4665	0.1115	0.7778	1.0907	1.0660	0.9417	0.5925	0.1753	0.0234

Table 6.1: Percentage relative error of MC simulation at different times $\tau = 0.25, 0.5, 0.75, 1$.

According to the table above, the percentage relative error varies widely depending on the values of v and τ. In some cases, the error is very low (e.g., less than 1%), while in others, it is very high (e.g., over 50%). It can be seen that the smallest percentage relative error in the experiment is 0.0080, while the largest is 53.3146 if N_p and τ are inappropriate. These two values are quite different so be mindful when selecting the parameters.

Generally, it is well known that the MC approach errors tend to increase if run with a lower N_p but higher τ, which is evidently consistent with our experimental table. Additionally, there is also a note for selecting v that can affect the errors. For example, if $v = 4$ in the case of $\tau = 0.75$ or $\tau = 1$, the errors are significantly large, followed by $v = 5$. However, if we consider $\tau = 0.25$ or $\tau = 0.5$ with $v = 8$, there are the apparent occurrence of large errors, even with large numbers of N_p runs.

However, we cannot guarantee that every test running the error will be the same value every time because the MC method is a simulation based on Euler–Maruyama estimation with standard normal random variables. This means, we cannot ensure results and errors. This inconsistency is another of the MC approach's drawbacks.

The general overview here is that although the MC simulations can estimate the first conditional moment of the Heston-CEV hybrid model, the results obtained are not accurate because some percentage of relative errors are quite large. Another disadvantage of using the MC simulations is that it is relatively time-consuming, which we will discuss in the next subsection.

6.2.2 Computational times

Moving on to the consumption of computational time, the table for the time calculation of the first conditional moment from our closed-form formula (5.1) and the MC simulations for different values of N_p and τ is shown as table 6.2. This demonstration represents the effectiveness of our method in comparison to the MC simulations determined with the same parameters as the previous experiment.

Computational Times (s)					
	N_p	$\tau = 0.25\,s$	$\tau = 0.5\,s$	$\tau = 0.75\,s$	$\tau = 1.0s$
MC	5,000	7.3384	7.3090	7.3121	7.2752
	10,000	16.8115	16.3837	15.3887	15.4236
	20,000	29.2205	28.3195	27.7989	27.6303
	100,000	105.3112	104.0656	109.9809	106.6475
Exact		0.3599	0.0176	0.0079	0.0038

Table 6.2: Computational time(s) for MC simulation at different times $\tau = 0.25, 0.5, 0.75, 1$.

Overall, our approach demonstrates superior performance than the MC method, even when N_p and τ are selected.

The smallest computational time using the MC simulations is $7.2752\,s$ where $N_p = 5,000$ and $\tau = 1.0\,s$. In contrast, $0.0038\,s$ is consumed when using our implementation. The computation time is reduced by about 1,900 times.

Additionally, if we consider the largest error case of the MC method, which runs on $N_p = 100,000$ and $\tau = 0.75\,s$, the time consumed is $109.9809\,s$. In this case, it takes 13,900 times more time consumption compared to the time spent by our method at the same τ value. Another interesting thing is that the time taken when selecting $\tau = 0.25\,s$ is consumed significantly for the MC method.

From this chapter, it can be concluded that our approach can indeed reduce the computational time by using the MC simulations. Furthermore, our results are the exact solutions. In the next chapter, we will sum up all of our work, including the presentation of our potential future work.

CHAPTER VII

CONCLUSION, DISCUSSION AND FUTURE WORK

To conclude, in Chapter 1, we justified the motivation of this thesis based on the available literature review and provided the objective of our research. In Chapter 2, we studied the basic knowledge related to our work, including the stochastic process, the CIR process and the ECIR process, the CEV model, the Mash-Rosenfeld process, the Heston model, the simulation schemes for SDEs, the Feynman-Kac theorem, the Cholesky decomposition method, the Riccati equation, and the Bell polynomials. After that, we presented the definition of the extended Heston-CEV hybrid model in Chapter 3.

In Chapter 4, we demonstrated an analytical formula for the conditional moment of the extended Heston-CEV hybrid model (3.5), which is given in Theorem 4.1. In particular, the analytical formula for the constant parameters case is simplified and expressed in closed form in Theorem 4.2. The well-known class of the Heston model has been mentioned, see Corollary 1, and the closed-form formulas for conditional moments of S_t and x_t have been observed respectively, see Corollaries 2 and 3.

In Chapter 5, we showed applications of our analytical form formula for deriving simple closed-form formulas for some essential properties of probability as the conditional moments, the fractional conditional volatility, variance moments, mixed moments, covariance, and correlation.

In the context of the validations, in Chapter 6, we compared the solutions generated from our closed-form formula and the approximated results with those obtained from the MC simulations. From the experiment, it is noticeable that our closed-form formulas perfectly agree with the MC simulations. Nevertheless, the highlight of our approach is decreasing computational time consumption compared with the MC simulations, despite gaining the exact value. For the above reasons, our approach is evidently better than the MC approach.

For potential future work, we are interested in applying our approach to financial forward contracts, such as the variance swap contract by using some significant statistical probability, including the first and second conditional moment, the conditional volatility, and the conditional mixed moment. Furthermore, we would like to extend the two dimensions of the extended Heston-CEV hybrid model to become a three dimension process as the extended Heston-CEV hybrid model together with the stochastic interest rate process. Moreover, if we can successfully expand the model, we will apply this model to the European options.

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APPENDICES

In the appendix, we will provide the proof of PDE (4.2) and examples of mathematical programming codes used in this research, which are Mathematica and MATLAB.

The mathematical codes are used to find the analytical form formula of conditional moments and the MATLAB codes are related to the calculation and the validation of the accuracy of our analytical formulas with the MC simulations.

APPENDIX A:

According to the expression (4.2)

$$\begin{aligned} -U_\tau + \left(\mu(t) - \frac{v_t^{\frac{1}{\delta}}}{2}\right) U_x - \left(a(t)\left(b(t)v_t^{\frac{\delta-1}{\delta}} - v_t\right)\right) U_v \\ + \tfrac{1}{2}\left(v_t^{\frac{1}{2\delta}} U_{xx} + \sigma^2(t) v_t^{\frac{2\delta-1}{\delta}} U_{vv}\right) + \rho\sigma v_t U_{xv} = 0, \end{aligned} \tag{1}$$

where $U(x, v, A, B, \tau) = \mathbf{E}_t^{\mathbb{Q}}\left[e^{Ax_T + Bv_T^{\frac{1}{\delta}}}\right]$, the proof appears as below [22]:

Proof. Assume the N-dimensional of SDE to be as follows

$$\begin{aligned} dx_{1t} &= \mu_1\, dt + \sigma_{11}\, dZ_{1t} + \sigma_{12}\, dZ_{2t} + \cdots + \sigma_{1N}\, dZ_{Nt}, \\ dx_{2t} &= \mu_2\, dt + \sigma_{21}\, dZ_{1t} + \sigma_{22}\, dZ_{2t} + \cdots + \sigma_{2m}\, dZ_{mt} + \cdots + \sigma_{2N}\, dZ_{Nt}, \\ &\vdots \\ dx_{nt} &= \mu_n\, dt + \sigma_{n1}\, dZ_{1t} + \sigma_{n2}\, dZ_{2t} + \cdots + \sigma_{nm}\, dZ_{mt} + \cdots + \sigma_{nN}\, dZ_{Nt}, \\ &\vdots \\ dx_{Nt} &= \mu_N\, dt + \sigma_{N1}\, dZ_{1t} + \sigma_{N2}\, dZ_{2t} + \cdots + \sigma_{Nm}\, dZ_{mt} + \cdots + \sigma_{NN}\, dZ_{Nt}. \end{aligned}$$

By applying the N-dimensional Feynman-Kac theorem (2.16) with $N = 2$, the corresponding PDE can be presented as

$$\frac{\partial U}{\partial t} + \mu_1 U_x + \mu_2 U_v + \frac{1}{2}\left(\gamma_{11}\frac{\partial^2 U}{\partial x^2} + \gamma_{12}\frac{\partial^2 U}{\partial x \partial v} + \gamma_{21}\frac{\partial^2 U}{\partial v \partial x} + \gamma_{22}\frac{\partial^2 U}{\partial v^2}\right) = 0.$$

As system (2.12),

$$\mu_1 = \left(\mu(t) - \frac{1}{2}\, v_t^{\frac{1}{\delta}}\right),$$
$$\mu_2 = a(t)\left(b(t)\, v_t^{\frac{\delta-1}{\delta}} - v_t\right)$$
$$\sigma_{11} = v_t^{\frac{1}{2\delta}}$$
$$\sigma_{21} = \rho\,\sigma(t)\, v_t^{\frac{2\delta-1}{2\delta}} \text{ and}$$
$$\sigma_{22} = \sigma(t)\,\sqrt{1-\rho^2}\, v_t^{\frac{2\delta-1}{2\delta}}.$$

Then,

$$\frac{\partial U}{\partial t} + \mu_1 U_x + \mu_2 U_v + \frac{1}{2}\left(\sigma_{11}^2\,\frac{\partial^2 U}{\partial x^2} + \sigma_{11}\sigma_{21}\,\frac{\partial^2 U}{\partial x \partial v} + \sigma_{11}\sigma_{21}\,\frac{\partial^2 U}{\partial v \partial x} + (\sigma_{21}^2 + \sigma_{22}^2)\,\frac{\partial^2 U}{\partial v^2}\right) = 0.$$

Hence,

$$U_t + \left(\mu(t) - \frac{v_t^{\frac{1}{\delta}}}{2}\right) U_x + \left(a(t)\,(b(t) v_t^{\frac{\delta-1}{\delta}} - v_t)\right) U_v$$
$$+\tfrac{1}{2}\left(v_t^{\frac{1}{2\delta}} U_{xx} + \sigma^2(t) v_t^{\frac{2\delta-1}{\delta}}\, U_{vv}\right) + \rho\sigma v_t\, U_{xv} = 0. \qquad (2)$$

Since $\tau = T - t, d\tau = -dt$. Therefore, U is the solution of the following PDE

$$-U_\tau + \left(\mu(t) - \frac{v_t^{\frac{1}{\delta}}}{2}\right) U_x - \left(a(t)\,(b(t) v_t^{\frac{\delta-1}{\delta}} - v_t)\right) U_v$$
$$+\tfrac{1}{2}\left(v_t^{\frac{1}{2\delta}} U_{xx} + \sigma^2(t) v_t^{\frac{2\delta-1}{\delta}}\, U_{vv}\right) + \rho\sigma v_t\, U_{xv} = 0. \qquad (3)$$

□

APPENDIX B: Mathematica

The first moment: The Mathematica code to derive the closed-form formula for the first conditional moment.

$\theta -> \phi, \phi -> \theta$

$$A[\tau\_, \alpha\_] := \alpha$$

$$\theta = \frac{1}{\delta}\sqrt{(\rho\,\sigma\,\alpha - \kappa)^2 + \sigma^2\,(\alpha - \alpha^2)};$$

$$B[\tau\_, \alpha\_] := -\frac{(\alpha - \alpha^2)\,(e^{\theta\,\tau} - 1) - \left(\theta\,(e^{\theta\,\tau} + 1) + \left(\frac{\rho\,\sigma\,\alpha - \kappa}{\delta}\right)(e^{\theta\,\tau} - 1)\right)\beta}{\theta\,(e^{\theta\,\tau} + 1) - \left(\frac{\rho\,\sigma\,\alpha - \kappa}{\delta} - \frac{\sigma^2}{\delta^2}\,\beta\right)(e^{\theta\,\tau} - 1)}$$

$$d[\tau\_, \alpha\_] := \frac{2\,\delta^2}{\sigma^2}\,\mathrm{Log}\left[\frac{2\,\theta\,e^{\frac{\theta - \frac{\rho\,\sigma\,\alpha - \kappa}{\delta}}{2}\,\tau}}{\theta\,(e^{\theta\,\tau} + 1) - \left(\frac{\rho\,\sigma\,\alpha - \kappa}{\delta} - \frac{\sigma^2}{\delta^2}\,\beta\right)(e^{\theta\,\tau} - 1)}\right]$$

$$c[\tau\_, \alpha\_] := r\,\alpha\,\tau + \left(\frac{\kappa\,\phi}{\delta} + \frac{\sigma^2\,(1 - \delta)}{2\,\delta^2}\right) d[\tau, \alpha]$$

In[40]:= $$K[\tau\_, \alpha\_] := e^{A[\tau,\alpha]\,x + B[\tau,\alpha]\,v^{\frac{1}{\delta}} + c[\tau,\alpha]}$$

$$f[\tau\_, \alpha\_] := A[\tau, \alpha]\,x + B[\tau, \alpha]\,v^{\frac{1}{\delta}} + c[\tau, \alpha]$$

U_α

In[45]:= $$X1 = \mathrm{FullSimplify}[D[K[\tau, \alpha], \alpha] /.\ \alpha \to 0 /.\ \beta \to 0, \kappa > 0]$$

Out[45]= $$\frac{1}{4\,\delta\,\kappa^2}\,e^{-\frac{\kappa\,\tau}{\delta}}\left(-2\left(-1 + e^{\frac{\kappa\,\tau}{\delta}}\right) v^{\frac{1}{\delta}}\,\delta^2\,\kappa + \delta\left((-1 + \delta)\,\sigma^2 - 2\,\delta\,\kappa\,\phi\right) + e^{\frac{\kappa\,\tau}{\delta}}\left(4\,x\,\delta\,\kappa^2 - \kappa\,\sigma^2\,\tau + \delta\,\sigma^2\,(1 + \kappa\,\tau) + 2\,\delta\,\kappa^2\,\tau\,(2\,r - \phi) - \delta^2\left(\sigma^2 - 2\,\kappa\,\phi\right)\right)\right)$$

The second moment: The Mathematica code to derive the closed-form formula for the second conditional moment.

In[∘]:=
$$A[\tau\_,\ \beta\_] := \alpha$$

$$\theta = \frac{1}{\delta}\sqrt{(\rho\,\sigma\,\alpha - \kappa)^2 + \sigma^2\left(\alpha - \alpha^2\right)};$$

$$B[\tau\_,\ \beta\_] := -\frac{\left(\alpha - \alpha^2\right)\left(e^{\theta\,\tau} - 1\right) - \left(\theta\left(e^{\theta\,\tau} + 1\right) + \left(\frac{\rho\,\sigma\,\alpha - \kappa}{\delta}\right)\left(e^{\theta\,\tau} - 1\right)\right)\beta}{\theta\left(e^{\theta\,\tau} + 1\right) - \left(\frac{\rho\,\sigma\,\alpha - \kappa}{\delta} - \frac{\sigma^2}{\delta^2}\beta\right)\left(e^{\theta\,\tau} - 1\right)}$$

$$d[\tau\_,\ \beta\_] := \frac{2\,\delta^2}{\sigma^2}\,\mathrm{Log}\left[\frac{2\,\theta\,e^{\frac{\theta - \frac{\rho\,\sigma\,\alpha - \kappa}{\delta}}{2}\tau}}{\theta\left(e^{\theta\,\tau} + 1\right) - \left(\frac{\rho\,\sigma\,\alpha - \kappa}{\delta} - \frac{\sigma^2}{\delta^2}\beta\right)\left(e^{\theta\,\tau} - 1\right)}\right]$$

$$c[\tau\_,\ \beta\_] := r\,\alpha\,\tau + \left(\frac{\kappa\,\phi}{\delta} + \frac{\sigma^2\,(1 - \delta)}{2\,\delta^2}\right) d[\tau,\ \beta]$$

$$K[\tau\_,\ \beta\_] := e^{A[\tau,\beta]\,x + B[\tau,\beta]\,v^{\frac{1}{\delta}} + c[\tau,\beta]}$$

$$f[\tau\_,\ \beta\_] := A[\tau,\ \beta]\,x + B[\tau,\ \beta]\,v^{\frac{1}{\delta}} + c[\tau,\ \beta]$$

f_β

In[∘]:=
```
FullSimplify[D[f[τ, β], β] /. β → 0 /. α → 0 , κ > 0]
```

Out[∘]=
$$\frac{e^{-\frac{\kappa\,\tau}{\delta}}\left(2\,v^{\frac{1}{\delta}}\,\delta\,\kappa + \left(-1 + e^{\frac{\kappa\,\tau}{\delta}}\right)\left((-1 + \delta)\,\sigma^2 - 2\,\delta\,\kappa\,\phi\right)\right)}{2\,\delta\,\kappa}$$

U_β

In[∘]:=
```
FullSimplify[D[K[τ, β], β] /. β → 0 /. α → 0 , κ > 0]
```

Out[∘]=
$$\frac{e^{-\frac{\kappa\,\tau}{\delta}}\left(2\,v^{\frac{1}{\delta}}\,\delta\,\kappa + \left(-1 + e^{\frac{\kappa\,\tau}{\delta}}\right)\left((-1 + \delta)\,\sigma^2 - 2\,\delta\,\kappa\,\phi\right)\right)}{2\,\delta\,\kappa}$$

The mix moment: The Mathematica code to derive the closed-form formula for the mix conditional moment.

In[ ]:=
$$A[\tau\_, \alpha\_, \beta\_] := \alpha$$
$$\theta = \frac{1}{\delta}\sqrt{(\rho\,\sigma\,\alpha - \kappa)^2 + \sigma^2\,(\alpha - \alpha^2)};$$
$$B[\tau\_, \alpha\_, \beta\_] := -\frac{(\alpha - \alpha^2)\,(e^{\theta\tau} - 1) - \left(\theta\,(e^{\theta\tau} + 1) + \left(\frac{\rho\,\sigma\,\alpha - \kappa}{\delta}\right)(e^{\theta\tau} - 1)\right)\beta}{\theta\,(e^{\theta\tau} + 1) - \left(\frac{\rho\,\sigma\,\alpha - \kappa}{\delta} - \frac{\sigma^2}{\delta^2}\beta\right)(e^{\theta\tau} - 1)}$$
$$c[\tau\_, \alpha\_, \beta\_] := r\,\alpha\,\tau + \left(\frac{\kappa\,\phi}{\delta} + \frac{\sigma^2\,(1-\delta)}{2\,\delta^2}\right)\left(\frac{2\,\delta^2}{\sigma^2}\,\text{Log}\left[\frac{2\,\theta\,e^{\frac{\theta - \frac{\rho\,\sigma\,\alpha-\kappa}{\delta}}{2}\tau}}{\theta\,(e^{\theta\tau} + 1) - \left(\frac{\rho\,\sigma\,\alpha - \kappa}{\delta} - \frac{\sigma^2}{\delta^2}\beta\right)(e^{\theta\tau} - 1)}\right]\right)$$
$$K[\tau\_, \alpha\_, \beta\_] := e^{A[\tau,\alpha,\beta]\,x + B[\tau,\alpha,\beta]\,v^{\frac{1}{\delta}} + c[\tau,\alpha,\beta]}$$

$A_\alpha, B_\alpha, C_\alpha$

In[ ]:=
```
FullSimplify[D[A[τ, α, β], α] /. α → 0 /. β → 0 , κ > 0]

FullSimplify[D[B[τ, α, β], α] /. α → 0 /. β → 0 , κ > 0]

FullSimplify[D[c[τ, α, β], α] /. α → 0 /. β → 0 , κ > 0]
```

Out[ ]= 1

Out[ ]= $$\frac{\left(-1 + e^{-\frac{\kappa\tau}{\delta}}\right)\delta}{2\,\kappa}$$

Out[ ]= $$r\,\tau + \frac{e^{-\frac{\kappa\tau}{\delta}}\left(-\delta + e^{\frac{\kappa\tau}{\delta}}\,(\delta - \kappa\,\tau)\right)\left(-\left((-1+\delta)\,\sigma^2\right) + 2\,\delta\,\kappa\,\phi\right)}{4\,\delta\,\kappa^2}$$

$A_\beta, B_\beta, C_\beta$

In[ ]:=
```
FullSimplify[D[A[τ, α, β], β] /. α → 0 /. β → 0 , κ > 0]

FullSimplify[D[B[τ, α, β], β] /. α → 0 /. β → 0 , κ > 0]

FullSimplify[D[c[τ, α, β], β] /. α → 0 /. β → 0 , κ > 0]
```

Out[ ]= 0

Out[ ]= $e^{-\frac{\kappa\tau}{\delta}}$

Out[ ]= $$\frac{e^{-\frac{\kappa\tau}{\delta}}\left(-1 + e^{\frac{\kappa\tau}{\delta}}\right)\left((-1+\delta)\,\sigma^2 - 2\,\delta\,\kappa\,\phi\right)}{2\,\delta\,\kappa}$$

$A_{\alpha\beta}, B_{\alpha\beta}, C_{\alpha\beta}$

In[ ]:=
```
FullSimplify[D[A[τ, α, β], {α, 1}, {β, 1}] /. α → 0 /. β → 0 , κ > 0]
FullSimplify[D[B[τ, α, β], {α, 1}, {β, 1}] /. α → 0 /. β → 0 , κ > 0]
FullSimplify[D[c[τ, α, β], {α, 1}, {β, 1}] /. α → 0 /. β → 0 , κ > 0]
p[τ_, α_, β_] := FullSimplify[D[A[τ, α, β], {α, 1}, {β, 1}] /. α → 0 /. β → 0 , κ > 0]
q[τ_, α_, β_] := FullSimplify[D[B[τ, α, β], {α, 1}, {β, 1}] /. α → 0 /. β → 0 , κ > 0]
r[τ_, α_, β_] := FullSimplify[D[c[τ, α, β], {α, 1}, {β, 1}] /. α → 0 /. β → 0 , κ > 0]
```

Out[ ]= 0

Out[ ]= $$\frac{e^{-\frac{\kappa\tau}{\delta}}\,\sigma\left(\left(-1 + e^{\frac{\kappa\tau}{\delta}}\right)\delta\,\sigma + \kappa\,(2\,\kappa\,\rho - \sigma)\,\tau\right)}{2\,\delta\,\kappa^2}$$

Out[ ]= $$-\frac{e^{-\frac{\kappa\tau}{\delta}}\,\sigma\left(-\left((-1+\delta)\,\sigma^2\right) + 2\,\delta\,\kappa\,\phi\right)\left(-2\,\delta\,\kappa\,\rho + \kappa\,(-2\,\kappa\,\rho + \sigma)\,\tau + 2\,\delta\,\kappa\,\rho\,\text{Cosh}\left[\frac{\kappa\tau}{\delta}\right] + \delta\,(2\,\kappa\,\rho - \sigma)\,\text{Sinh}\left[\frac{\kappa\tau}{\delta}\right]\right)}{4\,\delta^2\,\kappa^3}$$

$f_{\alpha\beta}$

In[ ]:= $$\text{FullSimplify}\left[x * p[\tau, \alpha, \beta] + v^{\frac{1}{\delta}} * q[\tau, \alpha, \beta] + r[\tau, \alpha, \beta]\right]$$

Out[ ]= $$\frac{1}{4\,\delta^2\,\kappa^3}\,e^{-\frac{\kappa\tau}{\delta}}\,\sigma\left(2\,v^{\frac{1}{\delta}}\,\delta\,\kappa\left(\left(-1 + e^{\frac{\kappa\tau}{\delta}}\right)\delta\,\sigma + \kappa\,(2\,\kappa\,\rho - \sigma)\,\tau\right) - \left(-\left((-1+\delta)\,\sigma^2\right) + 2\,\delta\,\kappa\,\phi\right)\left(-2\,\delta\,\kappa\,\rho + \kappa\,(-2\,\kappa\,\rho + \sigma)\,\tau + 2\,\delta\,\kappa\,\rho\,\text{Cosh}\left[\frac{\kappa\tau}{\delta}\right] + \delta\,(2\,\kappa\,\rho - \sigma)\,\text{Sinh}\left[\frac{\kappa\tau}{\delta}\right]\right)\right)$$

APPENDIX C: MATLAB

The first moment: The MATLAB code for calculating and comparing the first conditional moment from our closed-form formula with the MC simulation, which is simulated using $\mu = 0.01$, $a = 0.1$, $\sigma = 0.001$, $\rho = 0.01$, $b = 0.1$, and $\delta = 1$, running for 5,000 sample paths.

```
% Code : The validation of the accuracy
clear all
clc

x0 = 2;
v0 = 1:10;

t = 0;
T = [1 2 3 4]*0.25;

r = 0.01;
k = 0.1; %kappa
sig = 0.001;
rho = 0.01;
the = 0.1;
%% -----------------------------------------------------
A11 = @(t) 1;
A10 = @(t) -1/(2*k)*(1-exp(-k*t));
A00 = @(t) -1/(2*k)*(the*exp(-k*t)-the-2*r*k*t+the*k*t);

for q = 1:length(T)
    tau = T(q)-t;
    for j = 1:length(v0)
        Uex(j) = A00(tau) + A10(tau)*v0(j) + A11(tau)*x0;
    end
    plot(v0,Uex,'LineWidth',1.5)
    hold on
end
%% ----------------------------------------
Ns = 5000;
Np = 5000;
E = zeros(4,length(v0));
for i = 1:length(T)
    tic
    T(i)
    tt = linspace(t,T(i),Ns);
    dt = tt(2)-tt(1);
    for j = 1:length(v0)
        x = x0*ones(Np,1);
        v = v0(j)*ones(Np,1);
        for q = 1:Ns-1
            dW1 = sqrt(dt)*randn(Np,1);
            dW2 = sqrt(dt)*randn(Np,1);
            x = x + (r-1/2*v)*dt + sqrt(v).*dW1;
            v = v + k*(the-v)*dt + rho*sig*sqrt(v).*dW1 + sig*sqrt(1-rho^2)*sqrt(v).*dW2;
        end
        Umc(j) = mean(x);
        %%% PERCENTAGE RELATIVE ERROR %%%
        E(i,j) = abs((Uex(j)-Umc(j))/Uex(j))*100;
    end
    plot(v0,Umc,'o','LineWidth',1.2)
    hold on
    toc
end
```

The second moment: The MATLAB code for calculating and comparing the second conditional moment from our closed-form formula with the MC simulation, which is simulated as the previous figure, running for 5,000 sample paths.

```
clear all
clc

x0 = 1;
v0 = 1:1:10;

t = 0;
T = [1 2 3 4]*0.25;

r = 0.01;
k = 0.1; %kappa
sig = 0.001;
rho = 0.01;
the = 0.1;

%% ----------------------------------------------------

A22 = @(t) 1;
A21 = @(t) -1/k*(1-exp(-k*t));
A20 = @(t) 1/(4*k^2)*exp(-2*k*t)*(exp(k*t)-1)^2;
A11 = @(t) -1/k*(the*exp(-k*t)-the-2*r*k*t+the*k*t);
A10 = @(t) 1/(4*k^3)*exp(-2*k*t)*(-2*k*the+4*k*the*exp(k*t)-2*k*the*exp(2*k*t)-4*k^2*exp(k*t) ...
            +4*k^2*exp(2*k*t)+4*r*k^2*t*exp(k*t)-4*r*k^2*t*exp(2*k*t)-2*the*k^2*t*exp(k*t)+2*the*k^2*t*exp(2*k*t) ...
            +4*rho*sig*k*exp(k*t)-4*rho*sig*k*exp(2*k*t)+4*rho*sig*k^2*t*exp(k*t)-sig^2+sig^2*exp(2*k*t) ...
            -2*sig^2*k*t*exp(k*t));
A00 = @(t) 1/(8*k^3)*exp(-2*k*t)*(2*the^2*k-4*the^2*k*exp(k*t)+2*the^2*k*exp(2*k*t) ...
            +8*the*k^2*exp(k*t)-8*the*k^2*exp(2*k*t)-8*r*the*k^2*t*exp(k*t) ...
            +8*r*the*k^2*t*exp(2*k*t)+4*the^2*k^2*t*exp(k*t)-4*the^2*k^2*t*exp(2*k*t) ...
            +8*r^2*k^3*t^2*exp(2*k*t)+8*the*k^3*t*exp(2*k*t) ...
            -8*r*the*k^3*t^2*exp(2*k*t)+2*the^2*k^3*t^2*exp(2*k*t) ...
            -16*the*rho*sig*k*exp(k*t)+16*the*rho*sig*k*exp(2*k*t) ...
            -8*the*rho*sig*k^2*t*exp(k*t)-8*the*rho*sig*k^2*t*exp(2*k*t) ...
            +the*sig^2+4*the*sig^2*exp(k*t)-5*the*sig^2*exp(2*k*t) ...
            +4*the*sig^2*k*t*exp(k*t)+2*the*sig^2*k*t*exp(2*k*t));

for q = 1:length(T)
    tau = T(q)-t;
    for j = 1:length(v0)
        Uex(j) = A00(tau) + A10(tau)*v0(j) + A11(tau)*x0 + A20(tau)*v0(j)^2 + A21(tau)*x0*v0(j) + A22(tau)*x0^2;
    end
    plot(v0,Uex,'LineWidth',1.5)
    hold on
end

%% ---------------------------------------

Ns = 5000;
Np = 5000;

for i = 1:length(T)
    tic
    T(i)
    tt = linspace(t,T(i),Ns);
    dt = tt(2)-tt(1);
%     pd = makedist('Normal',0,sqrt(dt));
    for j = 1:length(v0)
        x = x0*ones(Np,1);
        v = v0(j)*ones(Np,1);
        for q = 1:Ns-1
            dW1 = sqrt(dt)*randn(Np,1);
            dW2 = sqrt(dt)*randn(Np,1);
%             dW1 = random(pd,Np,1);
%             dW2 = random(pd,Np,1);
            x = x + (r-1/2*v)*dt + sqrt(v).*dW1;
            v = v + k*(the-v)*dt + rho*sig*sqrt(v).*dW1 + sig*sqrt(1-rho^2)*sqrt(v).*dW2;
        end
        Umc(j) = mean(x.^2);

    end
    plot(v0,Umc,'o','LineWidth',1.2)
    hold on
    toc
end


grid on
grid minor

xlabel('$v$','Interpreter','latex')
ylabel('$u_{\alpha}^{(2)}(x,v,\tau)$','Interpreter','latex')
legend({'$\tau=0.25$','$\tau=0.50$','$\tau=0.75$','$\tau=1.00$','MC with $\tau=0.25$','MC with $\tau=0.50$','MC with $\tau=0.

hold off
```

The mix moment: The MATLAB code for calculating and comparing the mix conditional moment from our closed-form formula with the MC simulation, which is simulated as the previous figure, running for 5,000 sample paths.

```
clear all
clc

% delta = 1

x0 = 2;
v0 = 1:10;

t = 0;
T = [1 2 3 4]*0.25;

r = 0.01;
k = 0.1; %kappa
sig = 0.001;
rho = 0.01;
the = 0.1;

%% ----------------------------------------------------
A01 = @(t) 1/(2*k^2)*(sig*exp(-k*t)*(sig*(exp(k*t)-1)+k*t*(2*k*rho-sig)))
A02 = @(t) -1/(4*k^3)*sig*exp(-k*t)*(2*k*the)

A11 = @(t) 1;
A10 = @(t) -1/(2*k)*(1-exp(-k*t));
A00 = @(t) -1/(2*k)*(the*exp(-k*t)-the-2*r*k*t+the*k*t);

A1 = @(t) exp(-k*t);
A2 = @(t) (exp(-k*t)-1)*the;

for q = 1:length(T)
    tau = T(q)-t;
    for j = 1:length(v0)
        Uex(j) = (A01(tau)*v0(j)+A02(tau)) + (A00(tau)+A10(tau)*v0(j)+A11(tau)*x0) * (A1(tau)*v0(j)+A2(tau));
    end
    plot(v0,Uex,'LineWidth',1.5)
    hold on
end
%% ----------------------------------------

Ns = 5000;
Np = 5000;

for i = 1:length(T)
    tic
    T(i)
    tt = linspace(t,T(i),Ns);
    dt = tt(2)-tt(1);
    for j = 1:length(v0)
        x = x0*ones(Np,1);
        v = v0(j)*ones(Np,1);
        for q = 1:Ns-1
            dW1 = sqrt(dt)*randn(Np,1);
            dW2 = sqrt(dt)*randn(Np,1);
            x = x + (r-1/2*v)*dt + sqrt(v).*dW1;
            v = v + k*(the-v)*dt + rho*sig*sqrt(v).*dW1 + sig*sqrt(1-rho^2)*sqrt(v).*dW2;
        end
        Umc(j) = mean(x.*v);
    end
    plot(v0,Umc,'o','LineWidth',1.2)
    hold on
    toc
end

grid on
grid minor

xlabel('$v$','Interpreter','latex')
ylabel('$u_{\alpha\beta}^{(1)}(x,v,\tau)$','Interpreter','latex')
legend({'$\tau=0.25$','$\tau=0.50$','$\tau=0.75$','$\tau=1.00$','MC with $\tau=0.25$','MC with $\tau=0.50$','MC with $\tau=0.

hold off
```

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