อินทิกรัลแบบเฮนสต็อก-สตีลต์เชสสำหรับฟังก์ชันซึ่งมีค่าอยู่ในปริภูมิ L^p

นายนพพร ธรรมรงค์รัตน์

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HENSTOCK-STIELTJES INTEGRAL FOR L^p-VALUED FUNCTIONS

Mr. Nopporn Thamrongrat

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By	Mr. Nopporn Thamrongrat
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Thesis Advisor	Associate Professor Imchit Termwuttipong, Ph.D.

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S. Hannongbua .... Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

Wicharn Lewkeeratiyutkul .... Chairman

(Associate Professor Wicharn Lewkeeratiyutkul, Ph.D.)

Imchit Termwuttipong .... Thesis Advisor

(Associate Professor Imchit Termwuttipong, Ph.D.)

.... Examiner

(Assistant Professor Songkiat Sumetkijakan, Ph.D.)

.... External Examiner

(Kittipong Laipaporn, Ph.D.)

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ในงานวิจัยนี้ เราให้นิยามของการอินทิเกรตแบบเฮนสต็อก-สตีลต์เชสสำหรับฟังก์ชันซึ่งมีค่าอยู่ในปริภูมิ L^p และศึกษาสมบัติของการอินทิเกรตดังกล่าว การศึกษาครั้งนี้ได้แนวคิดมาจากการอินทิเกรตแบบเฮนสต็อกสำหรับฟังก์ชันค่าจริง ซึ่งคิดค้นโดย J.Kurzweil และ R.Henstock

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In this thesis, we develop a Henstock-Stieltjes integral for L^p-valued functions and its theory. This work is based on the idea of Henstock integration introduced by J.Kurzweil and R.Henstock.

Department : ....Mathematics....	Student's Signature : ..Nopporn.....Thamrongrat
Field of Study : ....Mathematics....	Advisor's Signature : Imchit Termwuttipong
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CONTENTS

CHAPTER I

INTRODUCTION

In the theory of Riemann integration, a real-valued function f defined on $[a, b]$ is *Riemann integrable* [1] if there exists $A \in \mathbb{R}$ such that for each $\varepsilon > 0$, there is a constant $\delta > 0$ such that

$$|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A| < \varepsilon,$$

for any partition $P = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ of $[a, b]$ with $t_i \in [x_{i-1}, x_i]$ and $x_i - x_{i-1} < \delta$ for $1 \leq i \leq n$.

It is noted that if such a real number A exists, it is unique. We usually denote A by $\int_a^b f$, and we say that f is Riemann integrable on $[a, b]$ with the integral $\int_a^b f$.

It is well known that the idea of Riemann integration is intuitive and simple, but the defects in the Riemann integral are several. The most serious one is that the class of Riemann integrable functions is too small. That is many simple functions are not Riemann integrable. For example, the Dirichlet function which assigns the value 1 for rationals in $[0, 1]$ and 0 elsewhere in $[0, 1]$ is not Riemann integrable. Even if Lebesgue integration which was developed in the beginning of the twentieth century can be applied to a wider class of functions and its convergence properties is superior to the Riemann integral, the class of Lebesgue integrable functions is still not large enough and the computation is rather sophisticated. In fact, we desire not only to gain an integral that the class of integrable functions is large but also to get an integral in which every derivative is integrable that is not true for Lebesgue integral. For instance, the function $f : [0,1] \to \mathbb{R}$ defined by $f(t) = t^2 \cos(\frac{\pi}{t^2})$ for $t \neq 0$ and $f(0) = 0$ is differentiable on $[0,1]$ but not Lebesgue integrable. We will see later for a more detail in example 2.21.

Lately a new integration, Henstock integration or Gauge integration, was introduced by Jaroslav Kurzweil and Ralph Henstock in the year 1957. The idea of the Henstock integral is a tiny variation of the definition of the classical Riemann integral and no measure theory is required, the Henstock integral has corrected the defects in the classical Riemann theory and both simplifies and extends the Lebesgue theory of integration. Moreover, all derivatives are (Henstock) integrable.

In 1992, Sergio S. Cao [3] generalized the definition of the Henstock integral for real-valued functions to functions taking values in Banach spaces and investigated some of its properties. Six years later, J.S. Lim and his colleagues [9] extended the idea of the Henstock integral to the Henstock-Stieltjes integral for real-valued functions. In 1999, Jean-Christophe Feauveau [4] developed the properties of an integral (the generalized Henstock integral) for Banach-valued functions including the theory of convergence; Monotone Convergence Theorem (MCT) and Dominated Convergence Theorem (DCT), that S.S. Cao and J.S. Lim did not mention. Also there are works studied on the Henstock integral, such as [2],[5],[8].

The work of J.C. Feauveau exposed an interesting view that: If a Banach-valued function f defined on $[a,b]$ is integrable then there exists a unique vector A in the Banach space being the integral of f. The immediate question from Feauveau is how can we define Henstock-Stieltjes integral for functions whose values in Banach spaces. Unfortunately, the Stieltjes integral needs the idea of the product of two elements in Banach space but we do not have the concept of the product of vectors in abstract Banach space. However, in some function spaces, we can define the product.

The purpose of this research is to define Henstock-Stieltjes integral for L^p-valued functions with respect to L^q-valued functions and investigate some theory on it. Furthermore, this study extends and improves the work of Piyaporn Juhung [7] that studied in the special case that $p = q = 2$.

This work is arranged to three chapters. Chapter I is an introduction. Chapter II deals with the Henstock integration for real-valued functions that gives us the idea of the integral in later chapter. The main work of our research is in Chapter III.

CHAPTER II

PRELIMINARIES

2.1 Henstock Integral for Real-Valued Functions

In this section we summarize the Henstock integral for real-valued function. We sum up the motivation of the integral in the first part and give some properties in the second part. Lastly, we present some examples for clearing the idea of the integral.

2.1.1 Introduction to the Henstock Integral

We now recall the definition of Henstock integration that was introduced by J. Kurzweil, in 1957, and used in theory of ordinary differential equations. The integral was discovered independently by R. Henstock who studied the major convergence theorems (Monotone and Dominated Convergence Theorems) for the integral. For texts containing expositions of the integral, we advise the readers to see in [1], [2], [5], [8], and [11]. The definition of the Henstock integral is a tiny difference of the definition of the Riemann integral but the effect of the variation is deeply intricate. More precisely, the Henstock integral is more general than the Lebesgue integral, namely, the class of Henstock integrable functions contains the class of Lebesgue for the real-valued functions whose domains are closed interval in $\mathbb{R}$. Nevertheless, similar to the Lebesgue integral, the Henstock integral satisfies the convergence properties and the Fundamental Theorem of Calculus (FTC) holds in full generality.

We commence the idea of the definition of the Henstock integral by focusing on the FTC in that if $f : [a, b] \to \mathbb{R}$ is a function which has a derivative f' everywhere in $[a, b]$, then the desire from this statement is f' is integrable and

$$\int_a^b f' = f(b) - f(a). \tag{2.1}$$

Unfortunately, the FTC for the Riemann integral requires the assumption that the derivative f' is Riemann integrable while the Lebesgue integral requires the boundedness of f' on $[a, b]$. To obtain a new integral bringing about the statement (2.1), we consider the following helpful lemma called *The Straddle Lemma.*

Lemma 2.1. *(The Straddle Lemma) Let $f : [a, b] \to \mathbb{R}$ be differentiable at $z \in [a, b]$. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|f(v) - f(u) - f'(z)(v - u)| < \varepsilon(v - u)$$

whenever $u \le z \le v$ and $[u, v] \subset [a, b] \cap (z - \delta, z + \delta)$.

Proof. Let $\varepsilon > 0$. Since f is differentiable at $z \in [a, b]$, there is a $\delta > 0$ such that

$$|\frac{f(x) - f(z)}{x - z} - f'(z)| < \varepsilon$$

for $0 < |x - z| < \delta$, $x \in [a, b]$. Let $[u, v] \subset [a, b] \cap (z - \delta, z + \delta)$ and $z \in [u, v]$. The conclusion of the lemma is complete in the case of $z = u$ or $z = v$. Now, suppose that $u < z < v$. Then

$$\begin{aligned} |f(v) - f(u) - f'(z)(v - u)| &\le |f(v) - f(z) - f'(z)(v - z)| + |f(z) - f(u) - f'(z)(z - u)| \\ &< \varepsilon(v - z) + \varepsilon(z - u) = \varepsilon(v - u). \end{aligned}$$

□

The geometric explanation of the Straddle Lemma is that the slope of the tangent line at $(z, f(z))$ can be approximated by the slope of the chord between $(u, f(u))$and $(v, f(v))$.

This lemma gives us a hint of how to proceed the FTC in the statement (2.1). For the Henstock-Kurzweil integral, we are interested only in partitions $P = \{x_0, x_1, \ldots, x_n\}$ and the set of points $\{y_1, y_2, \ldots, y_n\}$ such that

$$[x_{i-1}, x_i] \subset (y_i - \delta(y_i), y_i + \delta(y_i)),$$

where δ is a positive function on $[a, b]$ resembling the condition of the Straddel Lemma. Before we complete the FTC, we will give a formal definition of the Henstock integral.

Definition 2.2. Let $[a, b]$ be a nondegenerate interval in $\mathbb{R}$. A *gauge* on $[a, b]$ is a positive function defined on $[a, b]$. For a partition $P = \{x_0, x_1, \ldots, x_n\}$ or $P = \{[x_{i-1}, x_i] \mid i = 1, 2, \ldots, n\}$ of $[a, b]$ and a subset $\{t_1, t_2, \ldots, t_n\}$ of $[a, b]$ with $t_{i-1} \le t_i$ for $i = 1, 2, \ldots, n$, the set

$$D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$$

is called a *tagged partition.* If δ is a gauge on $[a, b]$, a tagged partition D is said to be a *δ-fine partition* if for each $i = 1, 2, \ldots, n$,

$$t_i - \delta(t_i) < x_{i-1} \le t_i \le x_i < t_i + \delta(t_i).$$

The points $t_1, t_2, \ldots, t_n$ are sometimes called *tags* of the tagged partition D. A *subpartition* of $[a, b]$ we mean a collection $\{J_j \mid j = 1, 2, \ldots, s\}$ of nonoverlapping closed intervals in $[a, b]$.

If δ is a gauge on $[a, b]$, the collection $\{(J_j, t_j) \mid j = 1, 2, \ldots, s\}$ consisting of a subpartition $\{J_j \mid j = 1, 2, \ldots, s\}$ of $[a, b]$ and a subset $\{t_1, t_2, \ldots, t_n\}$ of $[a, b]$ with $t_{i-1} \le t_i$ for $i = 1, 2, \ldots, n$ is called a *δ-fine subpartition* of $[a, b]$ if for all $j = 1, 2, \ldots, s$,

$$t_j \in J_j \subset (t_j - \delta(t_j), t_j + \delta(t_j))$$

Let $f : [a, b] \to \mathbb{R}$, δ a gauge on $[a, b]$ and $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ a δ-fine partition of $[a, b]$. The Riemann sum of f on $[a, b]$ with respect to D is defined to be

$$S(f, D) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

Definition 2.3. Let $f : [a, b] \to \mathbb{R}$. We say that f is *Henstock integrable* on $[a, b]$ if there exists a real number A with the property that for every $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that for any δ-fine partition $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ of $[a, b]$, we have

$$|S(f, D) - A| < \varepsilon.$$

The number A in the definition is called the integral of f on $[a, b]$ and is denoted by $\int_{[a,b]} f$ or $\int_a^b f$.

Using the lemma 2.1, we directly derive the FTC in the statement (2.1) called the FTC part I for the Henstock integral.

Theorem 2.4. *(The FTC :Part I) If $f : [a, b] \to \mathbb{R}$ is a function which has a derivative f' everywhere in $[a, b]$, then f' is Henstock integrable and*

$$\int_a^b f' = f(b) - f(a).$$

It is obvious that f is Henstock integrable on $[a, b]$ implies f is Riemann integrable on $[a, b]$. More precisely the Henstock integral is defined in the same way of the Riemann integral as a limit except the δ-fineness of partition is measured by the gauge δ instead of a positive constant. The next lemma guarantees the existence of δ-fine partition of $[a, b]$ for each gauge δ on $[a, b]$. It is known as *The Consin's Lemma* . For the proof of it, see in [1].

Lemma 2.5. *(Cousin's Lemma) [5] If δ is a gauge on a closed and bounded interval $[a, b]$ and $[c, d]$ is any closed subinterval of $[a, b]$, then there always exists a δ-fine partition of $[c, d]$.*

2.1.2 Basic Properties of the Integral

The aim of this section is to give some properties of the Henstock integration that will be duplicated in later chapter.

Theorem 2.6. *Let $f : [a,b] \to \mathbb{R}$. Then f is Henstock integrable on $[a,b]$ if and only if for each positive real number ε, there exists a gauge δ on $[a,b]$ such that for any δ-fine partitions $D_1 = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ and $D_2 = \{([x'_{i-1}, x'_i], t'_i) \mid i = 1, 2, \ldots, m\}$ of $[a,b]$, we have*

$$|S(f, D_1) - S(f, D_2)| < \varepsilon.$$

Theorem 2.7. *Let $f, f_1, f_2 : [a,b] \to \mathbb{R}$ be Henstock integrable on $[a,b]$. Then,*

(*i*) *$f_1 + f_2$ is Henstock integrable on $[a,b]$ with $\int_a^b (f_1 + f_2) = \int_a^b f_1 + \int_a^b f_2$.*

(*ii*) *For every $\lambda \in \mathbb{R}$, λf is Henstock integrable on $[a,b]$ with $\int_a^b \lambda f = \lambda \int_a^b f$.*

(*iii*) *If $f \geq 0$ on $[a,b]$, then $\int_a^b f \geq 0$.*

(*iv*) *If $f_1 \geq f_2$ on $[a,b]$, then $\int_a^b f_1 \geq \int_a^b f_2$.*

We say that f is *absolutely Henstock integrable* on $[a,b]$ if both f and $|f|$ are integrable on $[a,b]$. We will see later in example 2.21. that the absolute integrability assumption is necessary for the next theorem.

Theorem 2.8. *If $f : [a,b] \to \mathbb{R}$ is absolutely Henstock integrable on $[a,b]$, then*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Theorem 2.9. *Let $f : [a,b] \to \mathbb{R}$ and $c \in [a,b]$. Then f is Henstock integrable on $[a,b]$ if and only if its restriction to $[a,c]$ and $[c,b]$ are both Henstock integrable. In this case, we have*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem 2.10. *If $f : [a,b] \to \mathbb{R}$ is Henstock integrable on $[a,b]$ and $[c,d] \subset [a,b]$, then it is Henstock integrable on $[c,d]$.*

The next theorem is known as *The Saks-Henstock Lemma*, it is fundamentally important in proving many properties of the Henstock integral.

Theorem 2.11. *(Saks-Henstock Lemma) Let $f : [a,b] \to \mathbb{R}$ be Henstock integrable on $[a,b]$. Then for each $\varepsilon > 0$ let δ be a gauge on $[a,b]$ such that if $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ is a δ-fine partition of $[a,b]$, then*

$$|S(f,D) - \int_a^b f| < \varepsilon.$$

If $D' = \{([x_i, y_i], t_i) \mid i = 1, 2, \ldots, n\}$ is any δ-fine subpartition of $[a,b]$, then

$$|\sum_{i=1}^{n}\{f(t_i)[y_i - x_i] - \int_{x_i}^{y_i} f\}| = |S(f,D') - \int_{\bigcup_{i=1}^{n}[x_i,y_i]} f| < \varepsilon.$$

Theorem 2.12. *(FTC: Part II) Let $f : [a,b] \to \mathbb{R}$ be Henstock integrable on $[a,b]$ and set $F(x) = \int_a^x f$ for $a \leq x \leq b$. If f is continuous at $x \in [a,b]$, then F is differentable at x with $F'(x) = f(x)$.*

Theorem 2.13. *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f is Henstock integrable on $[a,b]$.*

Theorem 2.14. [11] *If $f : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$, then f is Henstock integrable on $[a,b]$ and two integrals agree.*

This implies that the class of Henstock integrable function is bigger than that of Lebesgue. The example 2.21. displays the converse of this theorem is false.

Corollary 2.15. *If $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$, then f is Henstock integrable on $[a,b]$ and two integrals agree.*

Theorem 2.16. *(Integration by parts)* [8] *Let $F, G, f, g : [a,b] \to \mathbb{R}$. Suppose that F and G are continuous and $F' = f$ and $G' = g$ on $[a,b]$. Then $Fg + fG$ is Henstock integrable on $[a,b]$ and*

$$\int_a^b (Fg + fG) = F(b)G(b) - F(a)G(a).$$

A sequence $(f_k) : [a,b] \to \mathbb{R}$ is said to be *increasing* on $[a,b]$ if $f_k(x) \le f_{k+1}(x)$ for all $x \in [a,b], k \in \mathbb{N}$. It is said to be *decreasing* on $[a,b]$ if $f_k(x) \ge f_{k+1}(x)$ for all $x \in [a,b], k \in \mathbb{N}$. A sequence is said to be *monotone* on $[a,b]$ if it is either increasing on $[a,b]$ or decreasing on $[a,b]$.

Theorem 2.17. *(Monotone Convergence Theorem) Suppose that (f_k) is a monotone sequence of Henstock integrable functions on $[a,b]$ converging pointwise to f on $[a,b]$. Then f is Henstock integrable on $[a,b]$ if and only if the sequence $(\int_a^b f_k)$ is bounded on $[a,b]$. In this case,*

$$\lim_{k\to\infty} \int_a^b f_k = \int_a^b f.$$

Theorem 2.18. *(Dominated Convergence Theorem) Suppose (f_k) is a sequence of Henstock integrable functions on $[a,b]$ converging pointwise to f on $[a,b]$. If we have Henstock integrable functions g and h such that $g \le f \le h$ for all $k \in \mathbb{N}$, then f is Henstock integrable and*

$$\lim_{k\to\infty} \int_a^b f_k = \int_a^b f\,.$$

2.1.3 Some Examples

In this section, we give some examples of functions that are Henstock integrable but not Riemann integrable. These examples lead us to see clearly the idea of the integral.

Example 2.19. [5] Consider the function $f(x) = \frac{1}{\sqrt{x}}$ for $x \in (0,1]$ and $f(0) = 0$.

Let $\varepsilon > 0$. Assume that $\varepsilon < 3/4$. Define a gauge δ on $[0,1]$ by

$$\delta(x) = \begin{cases} \varepsilon x^2 & \text{if } \ 0 < x \le 1, \\ \varepsilon^2 & \text{if } \ x = 0. \end{cases}$$

Suppose that $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ is a δ-fine partition of $[0,1]$. We now show that 0 must be the tag t_1. Since D is a δ-fine partition of $[0,1]$,

$[0, x_1] \subset (t_1 - \delta(t_1), t_1 + \delta(t_1))$. This implies that $t_1 - \delta(t_1) < 0$. If $t_1 > 0$, then $\delta(t_1) = \varepsilon t_1^2$, so that $t_1 - \delta(t_1) = t_1 - \varepsilon t_1^2 > 0$ which is a contradiction. Therefore t_1 must be 0. Now consider $([x_{i-1}, x_i], t_i) \in D$ for $i > 1$. Since $x_{i-1} > t_i - \varepsilon t_i^2 \geq t_i - \varepsilon t_i > t_i/4$, $\sqrt{t_i}(\sqrt{t_i} + \sqrt{x_{i-1}})^2 > t_i^2$,

$$\sqrt{t_i}(\sqrt{t_i} + \sqrt{x_{i-1}})^2 > \sqrt{t_i}(\sqrt{t_i} + \sqrt{t_i/4})^2 = \frac{9}{4} t_i \sqrt{t_i} > t_i \sqrt{t_i} > t_i^2.$$

It follows that

$$\begin{aligned} \left|\frac{2\sqrt{t_i} - 2\sqrt{x_{i-1}}}{t_i - x_{i-1}} - \frac{1}{\sqrt{t_i}}\right| = \left|\frac{2}{\sqrt{t_i} + \sqrt{x_{i-1}}} - \frac{1}{\sqrt{t_i}}\right| &= \frac{\sqrt{t_i} - \sqrt{x_{i-1}}}{\sqrt{t_i}(\sqrt{t_i} + \sqrt{x_{i-1}})} \\ &= \frac{t_i - x_{i-1}}{\sqrt{t_i}(\sqrt{t_i} + \sqrt{x_{i-1}})^2} < \frac{\varepsilon t_i^2}{t_i^2} = \varepsilon. \end{aligned}$$

Similarly, it can be shown that

$$\left|\frac{2\sqrt{x_i} - 2\sqrt{t_i}}{x_i - t_i} - \frac{1}{\sqrt{t_i}}\right| < \varepsilon.$$

Combining these two inequalities yields

$$\begin{aligned} \left|\frac{1}{\sqrt{t_i}}(x_i - x_{i-1}) - 2(\sqrt{x_i} - \sqrt{x_{i-1}})\right| &\leq \left|\frac{1}{\sqrt{t_i}}(x_i - t_i) - 2(\sqrt{x_i} - \sqrt{t_i})\right| \\ &\quad + \left|\frac{1}{\sqrt{t_i}}(t_i - x_{i-1}) - 2(\sqrt{t_i} - \sqrt{x_{i-1}})\right| \\ &= \left|\frac{2(\sqrt{x_i} - \sqrt{t_i})}{x_i - t_i} - \frac{1}{\sqrt{t_i}}\right|(x_i - t_i) \\ &\quad + \left|\frac{2(\sqrt{t_i} - \sqrt{x_{i-1}})}{t_i - x_{i-1}} - \frac{1}{\sqrt{t_i}}\right|(t_i - x_{i-1}) \\ &< \varepsilon(x_i - t_i) + \varepsilon(t_i - x_{i-1}) = \varepsilon(x_i - x_{i-1}). \end{aligned}$$

Hence,

$$\begin{aligned} |S(f, D) - 2| &= \left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - \sum_{i=1}^{n}(2\sqrt{x_i} - 2\sqrt{x_{i-1}})\right| \\ &\leq 2\sqrt{x_1} + \sum_{i=2}^{n}\left|\frac{1}{\sqrt{t_i}}(x_i - x_{i-1}) - (2\sqrt{x_i} - 2\sqrt{x_{i-1}})\right| \\ &< 2\varepsilon + \sum_{i=2}^{n} \varepsilon(x_i - x_{i-1}) < 3\varepsilon. \end{aligned}$$

This shows that f is Henstock integrable with $\int_0^1 f = 2$.

The first example manifests the role of the gauge that can force one to take a particular point as a tag. This can be useful when a particular point is a source of difficulty.

Example 2.20. We consider the well known function called the Dirichlet function that is defined on $[0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Even if f is discontinuous at every point and it is not Riemann integrable, we will now show that the function is Henstock integrable with the integral 0.

Let $\{r_k : k \in \mathbb{N}\}$ be an enumeration of the rational numbers in $[0,1]$ and $\varepsilon > 0$. We define the gauge

$$\delta(t) = \begin{cases} \varepsilon/2^{k+1} & \text{if } t = r_k, \\ 1 & \text{if } t \text{ is irrational.} \end{cases}$$

Let $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ be a δ-fine partition of $[0,1]$. If the tag $t_i \in [x_{i-1}, x_i]$ is irrational, then $f(t_i) = 0$ and so $f(t_i)[x_i - x_{i-1}] = 0$. If the tag $t_i \in [x_{i-1}, x_i]$ is rational, then $f(t_i) = 1$ and if $t_i = r_k$ for some $k \in \mathbb{N}$, then $f(t_i)[x_i - x_{i-1}] = x_i - x_{i-1} < 2\delta(r_k) = \varepsilon/2^k$. Since only rational tags make a nonzero contribution to $S(f, D)$, we have

$$|S(f,D)| < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

This implies that the Dirichlet function is Henstock integrable with the integral zero.

The computation in Example 2.20 illustrates the advantage of using gauges with variable length. Even though the Dirichlet function is usually used to show that a function which is not Riemann integrable may be Henstock integrable, the function is Lebesgue integrable, as well. The next example demonstrates that a Henstock integrable function need not to be Lebesgue integrable. Moreover, this example attest that the Henstock integral is not absolutely integrable that is contrast to the situation for the Riemann and the Lebesgue integrals where the absolute value of an integrable function is also integrable.

Example 2.21. [8] For $0 < t \leq 1$, let $f(t) = t^2 \cos(\pi/t^2)$ and $f(0) = 0$.

Then f is differentiable on $[0, 1]$ with

$$f'(t) = \begin{cases} 0 & \text{if } \ t = 0 \\ 2t\cos(\pi/t^2) + \frac{2\pi}{t}\sin(\pi/t^2) & \text{if } \ 0 < t \leq 1. \end{cases}$$

By FTC part I, f' is Henstock integrable and $\int_0^1 f' = f(1) - f(0) = -1$. Next we will show that f' is not absolutely integrable on $[0, 1]$. It is clear that f' is continuous at every point of $(0, 1]$. Therefore $|f'|$ is integrable on every closed subinterval in $(0, 1]$. Setting $b_k = 1/\sqrt{2k}$ and $a_k = \sqrt{2/(4k+1)}$, we see that $\int_{a_k}^{b_k} f' = 1/2k$. Since the intervals $[a_k, b_k]$ are pairwise disjoint, if $|f'|$ is Henstock integrable on $[0, 1]$, we have

$$\sum_{k=1}^{n} \frac{1}{2k} \leq \sum_{k=1}^{n} |\int_{a_k}^{b_k} f'| \leq \int_0^1 |f'|$$

for all $n \in \mathbb{N}$. However, from the divergence of the harmonic series we infer that $|f'|$ is not Henstock integrable on $[0, 1]$. In the same way, it is not difficult to see that f' is not Lebesgue integrable. Recall that a function is Lebesque integrable if and only if its absolute value is Lebesgue integrable.

2.2 L^p-Spaces

In this section, we recall the definition of L^p-space and give some important properties used in the next chapter.

2.2.1 Definitions

Let (X, M, μ) be any measure space and let $1 \leq p < \infty$. We define a relation on $\mathfrak{L}^p := \{f : X \to \mathbb{C} \mid f$ is a measurable function and $\int_X |f|^p d\mu < \infty\}$ by $f \sim g$ if and only if $f = g$ almost everywhere. We can see that the relation above is an equivalence relation. Next, we define

$$L^p(X, \mu) = \{[f]_\sim \mid f \in \mathfrak{L}^p\}.$$

For $f, g \in L^p(X, \mu)$ and $c \in \mathbb{C}$, define

$$[f] + [g] = [f + g],\ c[f] = [cf] \text{ and } ||[f]||_p = (\textstyle\int_X |f|^p d\mu)^{1/p}.$$

We define metric on $L^p(X, \mu)$ by

$$d([f], [g]) = ||f - g||_p.$$

One can show that $L^p(X, \mu)$ is a Banach space. Hence, $L^p(X, \mu)$ is the space of all equivalence classes of functions in $\mathfrak{L}^p$ where two functions are in the same class if and only if they are equal almost everywhere, and we simply refer to $[f]$ in $L^p(X, \mu)$ by f.

For the case $p = \infty$. A measurable function f on X satisfying *ess sup*f := $\inf\{M \mid |f| \leq M$ *a.e.*$\} < \infty$ is said to be *essentially bounded* and we define $||f||_\infty$ to be

$$||f||_\infty = ess\ sup f.$$

We call $||f||_\infty$ the *essential supremum* of f. The space of all (equivalence classes of) essentially bounded measurable functions on (X, μ) will be denoted by $L^\infty(X, \mu)$. It is the same as the case $1 \leq p < \infty$, one can show that $L^\infty(X, \mu)$ is a Banach space.

Definition 2.22. Let (f_n) be a sequence in L^r and $f \in L^r$. We say that (f_n) *converges to* f in L^r if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$ implies $||f_n - f||_r < \varepsilon$. We write $\lim_{n\to\infty} f_n = f\,[L^r]$ to denote that (f_n) *converges to* f in L^r.

2.2.2 Some Properties of L^p-Spaces

The following properties are the basic tools for the next chapter. Precisely, the first two theorems is used to define a Henstock-Stieltjes integral for L^p-valued functions and the last lemma 2.25 is used to fulfill the Monotone Convergence Theorem.

Theorem 2.23. *(Hölder's inequality) Let p, q and r be positive real numbers satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p(X,\mu)$ and $g \in L^q(X,\mu)$, then $fg \in L^r(X,\mu)$ and*

$$||fg||_r \leq ||f||_p ||g||_q.$$

If we set $r = 1$ when $p = 1$ and $q = \infty$, then the above conclusion is still true.

Theorem 2.24. *(Minkowski's inequality) Let $1 \leq p \leq \infty$ and let f, g be non-negative measurable functions on a measure space (X,μ). Then*

$$||f+g||_p \leq ||f||_p + ||g||_p.$$

Lemma 2.25. *Let (f_n) be a sequence in $L^p(X,\mu)$ and $f \in L^p(X,\mu)$ where $p \geq 1$ and $\mu(X) < \infty$. If the sequence (f_n) converges to f in L^p and for each n, f_n is a non-negative function, then so is f.*

Proof. Let $A = \{x \in X \mid f(x) < 0\}$ and suppose $\mu(A) > 0$.
Let $A_n = \{x \in X \mid f(x) < -1/n\}$. Then $\bigcup_{n=1}^{\infty} A_n = A$. Since $\mu(A) > 0$, there is $n_0 \in \mathbb{N}$ such that $\mu(A_{n_0}) > 0$. Since (f_n) converges to f in L^p, there exists $N \in \mathbb{N}$ such that

$$\int_X |f_N - f|^p d\mu < (\frac{1}{n_0})^p \mu(A_{n_0}).$$

Let $B = \{x \in X \mid f_N(x) \geq 0\}$. Then $\mu(X \backslash B) = 0$ and hence

$$\mu(A_{n_0} \cap B) = \mu(A_{n_0}) > 0.$$

Thus $|f_N - f|^p \geq (\frac{1}{n_0})^p$ on $A_{n_0} \cap B$. Now ,

$$\begin{aligned}\int_X |f_N - f|^p d\mu &\geq \int_{A_{n_0} \cap B} |f_N - f|^p d\mu \\ &\geq (\frac{1}{n_0})^p \mu(A_{n_0} \cap B) \\ &= (\frac{1}{n_0})^p \mu(A_{n_0}), \qquad \text{a contradiction.}\end{aligned}$$

This shows that $\mu(A) = 0$, that is f is a non-negative function. □

CHAPTER III

Henstock-Stieltjes Integral for L^p-Valued Functions

In 2007, Juhung [7] studied the Henstock-Stieltjes integration of a function f whose value is in L^2-space with respect to an L^2-valued function g. In her work some interesting properties are not considered. It is a question that can we generalize the integration to the case that the functions $f(x) \in L^p$ and $g(x) \in L^q$.

In this chapter, we introduce a definition of Henstock-Stieltjes integral for L^p-valued functions with respect to L^q-valued functions and investigate some properties of the integral, and then the theory on the integral is established.

3.1 The Definition

Let (X, M, μ) be any measure space. To define a Henstock-Stieltjes integral for an L^p-valued function with respect to an L^q-valued function, the product of $f(x)$ and $g(x)$ are involved. By virtue of the theorem 2.23, a tool is provided. Throughout this chapter, we consider only the case that the positive real numbers p, q and $r \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. And for convenience, we will denote $L^p(X, \mu)$ and $L^q(X, \mu)$ shortly by L^p and L^q.

Definition 3.1. Let $f : [a, b] \to L^p$ and $g : [a, b] \to L^q$, we say that f is *Henstock-Stieltjes integrable with respect to* g on $[a, b]$ if for each $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that

$$|| \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})] - \sum_{i=1}^{m} f(d_i)[g(x'_i) - g(x'_{i-1})]||_r < \varepsilon,$$

for any two δ-fine partitions $D = \{([x_{i-1}, x_i], c_i) \mid i = 1, 2, \ldots, n\}$ and $D' = \{([x'_{i-1}, x'_i], d_i) \mid i = 1, 2, \ldots, m\}$ of $[a, b]$.

The above definition gives a criteria of integrability. Nevertheless, for an integrable function, the integral is not given. The next theorem provides a necessary and sufficient condition for a function f to be integrable with respect to a function g on $[a, b]$. Moreover, it states that the integral is an element in L^r. From now on, the statement "f is *Henstock -Stieltjes integrable with respect to* g on $[a, b]$" is stated in short by "f is *interable with respect to* g on $[a, b]$".

Theorem 3.2. *Let $f : [a, b] \to L^p$ and $g : [a, b] \to L^q$. The function f is integrable with respect to g on $[a, b]$ if and only if there exists a function $A \in L^r$ such that for each $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that whenever $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ is a δ-fine partition of $[a, b]$, we have*

$$||S_g(f, D) - A||_r < \varepsilon,$$

where $S_g(f, D) = \sum_{i=1}^{n} f(t_i)[g(x_i) - g(x_{i-1})]$.

Proof. Suppose that f is integrable with respect to g on $[a, b]$. For each positive integer n, choose a gauge δ_n on $[a, b]$ such that for any δ_n-fine partitions $D = \{([x_{i-1}, x_i], c_i) \mid i = 1, 2, \ldots, n\}$ and $D' = \{([x'_{i-1}, x'_i], d_i) \mid i = 1, 2, \ldots, m\}$ of $[a, b]$,

$$||S_g(f, D) - S_g(f, D')||_r < \frac{1}{n}. \tag{3.1}$$

We may assume that the sequence $(\delta_n)_{n\in\mathbb{N}}$ is non-increasing. For each $n \in \mathbb{N}$, let D_n be a fixed δ_n-fine partition. By the inequality (3.1), the sequence $(S_g(f, D_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in L^r. Let $A \in L^r$ be the limit of this sequence and let $\varepsilon > 0$. Choose a positive integer $N > 2/\varepsilon$ such that if $n \geq N$, then

$$||S_g(f, D_n) - A||_r < \frac{\varepsilon}{2}. \tag{3.2}$$

Let D'_N be a δ_N-fine partition. By the inequality (3.2), we have

$$\begin{aligned}||S_g(f, D'_N) - A||_r &\leq ||S_g(f, D'_N) - S_g(f, D_N)|| + ||S_g(f, D_N) - A||_r \\ &< \frac{1}{N} + \frac{\varepsilon}{2} \ < \ \varepsilon.\end{aligned}$$

To prove the converse, let $\varepsilon > 0$. There exists a gauge δ on $[a, b]$ such that whenever D is a δ-fine partition of $[a, b]$, we get

$$||S_g(f, D) - A||_r \ < \ \frac{\varepsilon}{2}.$$

Therefore, if D_1 and D_2 are δ-fine partitions of $[a, b]$, then

$$||S_g(f, D_1) - S_g(f, D_2)||_r \ < \ \varepsilon.$$

This implies that f is integrable on $[a, b]$. $\square$

The element A in the above theorem is called an integral of f with respect to g on $[a, b]$. It is obvious that if A_1 and A_2 are integrals of f with respect to g, then $A_1 = A_2$ a.e. in L^r. So the integral is unique if it exists, it is denoted by $\int_a^b f dg$.

3.2 Properties of the Integral

In this section, some properties of Henstock-Stieltjes integral are presented.

Theorem 3.3. *Let f_1 and f_2 be L^p-valued functions defined on $[a, b]$ and let $g : [a, b] \to L^q$. If f_1 and f_2 are integrable with respect to g on $[a, b]$, then the functions $f_1 + f_2$ and λf_1 are integrable with respect to g on $[a, b]$ for all $\lambda \in \mathbb{R}$, and*

$$\int_a^b (f_1 + f_2) dg = \int_a^b f_1 dg + \int_a^b f_2 dg,$$

$$\int_a^b \lambda f_1 dg = \lambda \int_a^b f_1 dg.$$

Proof. Let f_1 and f_2 be integrable with respect to g on $[a,b]$, and $\varepsilon > 0$. Let δ_1 and δ_2 be gauges on $[a,b]$ such that

$$||S_g(f_1, D') - \int_a^b f_1 dg||_r < \frac{\varepsilon}{2} \tag{3.3}$$

for any δ_1-fine partition D' of $[a,b]$ and

$$||S_g(f_2, D'') - \int_a^b f_2 dg||_r < \frac{\varepsilon}{2} \tag{3.4}$$

for any δ_2-fine partition D'' of $[a,b]$.

Let $\delta : [a,b] \to \mathbb{R}^+$ defined by $\delta(t) = \min\{\delta_1(t), \delta_2(t)\}$. Then δ is a gauge on $[a,b]$. For any δ-fine partition D of $[a,b]$, applying the triangle inequality to (3.3) and (3.4), we obtain

$$||S_g(f_1 + f_2, D) - (\int_a^b f_1 dg - \int_a^b f_2 dg)||_r < \ \varepsilon.$$

It is obvious that the integrability of f_1 implies the integrability of λf_1 for any $\lambda \in \mathbb{R}$ and

$$\int_a^b \lambda f_1 dg = \lambda \int_a^b f_1 dg.$$

□

Theorem 3.4. *Let $f : [a,b] \to L^p$, $g : [a,b] \to L^q$ and $\lambda \in \mathbb{R}$. If f is integrable with respect to g on $[a,b]$, then f is integrable with respect to λg on $[a,b]$. Furthermore,*

$$\int_a^b f d(\lambda g) = \int_a^b \lambda f dg = \lambda \int_a^b f dg.$$

Proof. Let $\varepsilon > 0$. There exists a gauge δ on $[a,b]$ such that

$$||S_g(f, D) - \int_a^b f dg||_r < \ \frac{\varepsilon}{|\lambda| + 1}$$

for any δ-fine partition D of $[a,b]$. Let D be a δ-fine partition of $[a,b]$. Then

$$||S_{\lambda g}(f, D) - \int_a^b \lambda f dg||_r = |\lambda| ||S_g(f, D) - \int_a^b f dg||_r.$$

It is clear that the integrability of f with respect to g implies that with respect to λg for any $\lambda \in \mathbb{R}$ and

$$\int_a^b fd(\lambda g) = \int_a^b \lambda f dg.$$

$\square$

Theorem 3.5. *If $f : [a,b] \to L^p$ is integrable with respect to both g_1 and $g_2 : [a,b] \to L^q$ on $[a,b]$, then f is integrable with respect to $g_1 + g_2$ on $[a,b]$ and*

$$\int_a^b fd(g_1 + g_2) = \int_a^b fdg_1 + \int_a^b fdg_2.$$

Proof. Let $\varepsilon > 0$ and δ_1, δ_2 be gauges on $[a,b]$ such that

$$||S_{g_1}(f, D') - \int_a^b fdg_1||_r < \frac{\varepsilon}{2} \tag{3.5}$$

for any δ_1-fine partition D' of $[a,b]$ and

$$||S_{g_2}(f, D'') - \int_a^b fdg_2||_r < \frac{\varepsilon}{2} \tag{3.6}$$

for any δ_2-fine partition D'' of $[a,b]$. Let $\delta : [a,b] \to \mathbb{R}^+$ be defined by $\delta(t) = \min\{\delta_1(t), \delta_2(t)\}$. Note that δ is a gauge on $[a,b]$. Let D be a δ-fine partition of $[a,b]$. It is easy to see that D is both δ_1-fine and δ_2-fine partition of $[a,b]$. Applying the triangle inequality to the inequalities (3.5) and (3.6), we then obtain

$$||S_{g_1+g_2}(f, D) - (\int_a^b fdg_1 + \int_a^b fdg_2)||_r < \varepsilon.$$

Since ε is arbitrary, the proof is now complete. $\square$

Theorem 3.6. *Let $c \in (a,b)$. Then $f : [a,b] \to L^p$ is integrable with respect to $g : [a,b] \to L^q$ on each of the intervals $[a,c]$ and $[c,b]$ if and only if f is integrable with respect to g on $[a,b]$. If this is the case, we have*

$$\int_a^b fdg = \int_a^c fdg + \int_c^b fdg.$$

Proof. Suppose that f is integrable with respect to g on each of the intervals $[a,c]$ and $[c,b]$. Let $\varepsilon > 0$. There exist positive functions δ_1 on $[a,c]$ and δ_2 on $[c,b]$ such that

$$||S_g(f,D') - \int_a^c fdg||_r < \frac{\varepsilon}{2} \text{ and } ||S_g(f,D'') - \int_c^b fdg||_r < \frac{\varepsilon}{2} \tag{3.7}$$

for any δ_1-fine partition D' of $[a,c]$ and δ_2-fine partition D'' of $[c,b]$. We define a gauge δ on $[a,b]$ by

$$\delta(t) = \begin{cases} \min\{\delta_1(t), \frac{1}{2}(c-t)\} & \text{if } t \in [a,c), \\ \min\{\delta_1(c), \delta_2(c)\} & \text{if } t = c, \\ \min\{\delta_2(t), \frac{1}{2}(c-t)\} & \text{if } t \in (c,b]. \end{cases}$$

Let $D = \{([x_{i-1},x_i],t_i) \mid i = 1,2,\ldots,n\}$ be any δ-fine partition of $[a,b]$. Then by the definition of δ, $c = t_{i_0}$ for some $i_0 \in \{1,2,\ldots,n\}$. Let $D' = \{([x_{i-1},x_i],t_i) \mid i = 1,\ldots,i_0-1\} \cup \{([x_{i_0-1},t_{i_0}],t_{i_0})\}$ and $D'' = \{([x_{i-1},x_i],t_i) \mid i = i_0+1,\ldots,n\} \cup \{([t_{i_0},x_{i_0}],t_{i_0})\}$. Then D' is a δ_1-fine partition of $[a,c]$ and D'' is a δ_2-fine partition of $[c,b]$. Since $S_g(f,D) = S_g(f,D') + S_g(f,D'')$ and from the inequality (3.7), we have

$$\begin{aligned} ||S_g(f,D) - (\int_a^c fdg + \int_c^b fdg)||_r &\leq ||S_g(f,D') - \int_a^c fdg||_r + ||S_g(f,D'') - \int_c^b fdg||_r \\ &< \varepsilon, \end{aligned}$$

and this implies that f is integrable on $[a,b]$ and $\int_a^b fdg = \int_a^c fdg + \int_c^b fdg$.

For the converse, we show only that f is integrable on $[a,c]$. Let $\varepsilon > 0$. Then there exists a gauge δ on $[a,b]$ such that for any two δ-fine partitions D' and D'' of $[a,b]$,

$$||S_g(f,D') - S_g(f,D'')||_r < \varepsilon. \tag{3.8}$$

Set $\delta' = \delta$ on $[a,c]$ and $\delta'' = \delta$ on $[c,b]$ and let D_1 and D_2 be any δ'-fine partitions of $[a,c]$ and choose a δ''-fine partition D_3 of $[a,b]$. Since both $D_1 \cup D_3$ and $D_2 \cup D_3$

are δ-fine partitions of $[a,b]$ and (3.8), we have

$$\begin{aligned}||S_g(f,D_1)-S_g(f,D_2)||_r &= ||S_g(f,D_1\cup D_3)-S_g(f,D_2\cup D_3)||_r\\ &< \varepsilon.\end{aligned}$$

This implies that f is integrable with respect to g on $[a,c]$. □

The next three corollaries are immediately obtained from the above theorem.

Corollary 3.7. *Let $[c,d]$ be a subinterval of $[a,b]$. If $f:[a,b]\to L^p$ is integrable with respect to $g:[a,b]\to L^q$ on $[a,b]$, then so is the restriction of f on $[c,d]$.*

Corollary 3.8. *If $f:[a,b]\to L^p$ is integrable with respect to $g:[a,b]\to L^q$ on $[a,b]$ and $a=c_0<c_1<\cdots<c_n=b$, then the restrictions of f on each subinterval $[c_{i-1},c_i]$ are integrable with respect to g and*

$$\int_a^b f dg = \sum_{i=1}^{n}\int_{c_{i-1}}^{c_i} f dg.$$

Definition 3.9. If $f:[a,b]\to L^p$ is integrable with respect to $g:[a,b]\to L^q$ on $[a,b]$ and $a\le c\le d\le b$, we define

$$\textstyle\int_d^c f dg = -\int_c^d f dg \ \text{ and } \ \int_c^c f dg = 0.$$

Corollary 3.10. *If $f:[a,b]\to L^p$ is integrable with respect to $g:[a,b]\to L^q$ on $[a,b]$ and c,d,e are any points in $[a,b]$, then*

$$\int_c^e f dg = \int_c^d f dg + \int_d^e f dg.$$

Theorem 3.11. *Let $f:[a,b]\to L^p$ and $g:[a,b]\to L^q$ be defined by $f(t)=h$ for all $t\in[a,b]$ and $h\in L^p$. Then f is integrable with respect to g on $[a,b]$ and*

$$\int_a^b f dg = h[g(b)-g(a)].$$

Proof. Here if $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \dots, n\}$ is any tagged partition of $[a, b]$, then

$$S_g(f, D) = \sum_{i=1}^{n} f(t_i)[g(x_i) - g(x_{i-1})] = h[g(b) - g(a)].$$

Let $\varepsilon > 0$ and a gauge δ defined by $\delta(x) = 1$ for all $x \in [a, b]$. Let $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \dots, n\}$ be a δ-fine partition of $[a, b]$. Then

$$||S_g(f, D) - h[g(b) - g(a)]||_r < \varepsilon.$$

Hence f is integrable with respect to g on $[a, b]$ and $\int_a^b f dg = h[g(b) - g(a)]$. $\square$

Next, we give two examples of Henstock-Stieltjes integrable functions.

Example 3.12. Define $f : [\frac{\pi}{2}, \pi] \to L^2(\mathbb{R}, \ell)$ and $g : [\frac{\pi}{2}, \pi] \to L^2(\mathbb{R}, \ell)$ by $f(t) = \chi_{[0,\pi]}$ and $g(t) = \chi_{[0,t]} + A(t)\cos t$ where

$$\chi_{[a,b]}(\tau) = \begin{cases} 1, & \tau \in [a, b] \\ 0, & \tau \in \mathbb{R} \backslash [a, b] \end{cases}$$

and $A(t)(\tau) = 1$ for all $t \in [\frac{\pi}{2}, \pi]$ and $\tau \in \mathbb{R}$.

By the theorem (3.11), we have that f is Henstock-Stieltjes integrable with respect to g on $[\frac{\pi}{2}, \pi]$ with $\int_{\frac{\pi}{2}}^{\pi} f dg = -\chi_{[0, \frac{\pi}{2}]}$.

Example 3.13. If $f(x) = 0$ for all $x \in [a, b]$ where 0 is the zero function in L^p, then $\int_a^b f dg = 0$.

In order to investigate the theory of the integration, we let the integrator g to be nice enough. It is obvious that the oscillation of g should not be too “big” in the following sense.

Definition 3.14. Let $g : [a, b] \to L^q$. We define the *variation of g over the interval* $I = [a, b]$ to be

$$Var(g, I) = \sup\{\sum_{i=1}^{n} ||g(x_i) - g(x_{i-1})||_q \mid P = \{x_0, x_1, \dots, x_n\}\}$$

where the supremum is taken over all partitions P of I. We say that g has(or is of) *bounded variation* on I if $Var(g, I) < \infty$. The collection of all functions defined on I with values in L^q that have bounded variation on I is denoted by $BV(I, L^q)$.

Definition 3.15. A function $g : [a, b] \to L^q$ is said to be a *Lipschitzian map* if there exists $\alpha \in [0, \infty)$ such that $||g(x) - g(y)||_q \leq \alpha |x - y|$ for all $x, y \in [a, b]$.

It is clear that if $g : [a, b] \to L^q$ is a Lipschitzian map, then g has bounded variation on $[a, b]$

Theorem 3.16. *Let $g : [a, b] \to L^q$ be a Lipschitzian map and $f : [a, b] \to L^p$ be the zero function almost everywhere on $[a, b]$. Then f is integrable with respect to g on $[a, b]$ and $\int_a^b f dg = 0$.*

Proof. Let $\alpha \in [0, \infty)$ be such that $||g(x) - g(y)||_q \leq \alpha |x - y|$ for all $x, y \in [a, b]$. Let $E = \{t \in [a, b] \mid f(t) \neq 0\}$ and for each $n \in \mathbb{N}$, let

$$E_n = \{t \in [a, b] \mid n - 1 \leq ||f(t)||_p < n\}.$$

Since $E = \bigcup_{n=1}^{\infty} E_n$ and E is of measure zero, so E_n is of measure zero for every n. Let $\varepsilon > 0$. For each n, let $\{J_{n,k}\}_{k=1}^{\infty}$ be a collection of open intervals such that

$$E_n \subseteq \bigcup_{k=1}^{\infty} J_{n,k} \text{ and } \sum_{k=1}^{\infty} \ell(J_{n,k}) < \frac{\varepsilon}{\alpha n 2^n} .$$

Define a gauge δ on $[a, b]$ as follows. For each $t \in [a, b]$ if $t \notin E$, let $\delta(t) = 1$; if $t \in E$, then $t \in E_m$ for some m and there is a $\delta_t > 0$ such that

$$(t - \delta_t, t + \delta_t) \subseteq \bigcup_{k=1}^{\infty} J_{m,k},$$

we define $\delta(t) = \delta_t$. Let $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, m\}$ be a δ-fine partition of $[a, b]$ and for each n, let D_n be the subset of D that has tags in E_n. By the definition of δ, every interval in D_n is a subset of $\bigcup_{k=1}^{\infty} J_{n,k}$. We now show that $||S_g(f, D)||_r < \ \varepsilon$.

For each δ-fine subpartition $D_n = \{([c_i, d_i], s_i) | i = 1, 2, \ldots, s\}$,

$$\begin{aligned} ||S_g(f, D_n)||_r &< \sum_{i=1}^{s} ||f(s_i)||_p ||g(d_i) - g(c_i)||_q \\ &< n(\sum_{i=1}^{s} \alpha(|d_i - c_i|)) \\ &\leq \alpha n \sum_{k=1}^{\infty} \ell(J_{n,k}) < \ \frac{\varepsilon}{2^n}. \end{aligned}$$

Since D_n's are pairwise disjoint and their union is D, the sum of the terms with tags in D is less than $\sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon$. Therefore $||S_g(f, D)||_r < \varepsilon$, and hence, f is integrable with respect to g on $[a, b]$ and $\int_a^b f dg = 0$. $\square$

Corollary 3.17. *If $f : [a, b] \to L^p$ is integrable with respect to a Lipschitzian map $g : [a, b] \to L^q$ on $[a, b]$ and $f = h$ almost everywhere on $[a, b]$, then h is integrable with respect to g on $[a, b]$ and*

$$\int_a^b f dg = \int_a^b h dg.$$

Proof. Since $f = h$ almost everywhere, $f - h = 0$ almost everywhere. Applying theorem 3.16, the function $f - h$ is integrable with respect to g on $[a, b]$ and $\int_a^b (f - h) dg = 0$. By linearity, the function $h = f + (h - f)$ is integrable and $\int_a^b h dg = \int_a^b f dg$. $\square$

Theorem 3.18. *Let $f : [a, b] \to L^p$ be a continuous function and let $g : [a, b] \to L^q$ be of bounded variation on $[a, b]$. Then f is integrable with respect to g on $[a, b]$.*

Proof. Let M be the variation of g over $[a, b]$ and $\varepsilon > 0$. Since f is uniformly continuous on $[a, b]$, there exists $\sigma > 0$ such that $||f(x) - f(y)||_p < \varepsilon$ for all $x, y \in [a, b]$ such that $|x - y| < 2\sigma$. Let N be a positive integer such that $\beta = \frac{b-a}{N} < \sigma$ and let $D_0 = \{([x_{i-1}, x_i], x_i) \mid i = 1, 2, \ldots, N\}$ where $x_j = a + j\beta$ for

$0 \leq j \leq N$. Define a gauge δ on $[a, b]$ by $\delta(x) = \frac{\beta}{2}$ for all $x \in [a, b]$. We will first show that

$$||S_g(f, D) - S_g(f, D_0)||_r < \varepsilon M$$

for any δ-fine partition D of $[a, b]$. Let $D = \{([y_{j-1}, y_j], t_j) \mid j = 1, 2, \ldots, N_1\}$ be a δ-fine partition of $[a, b]$. We define a set D_1 on $[a, b]$ from D as follows.

If (y_{j-1}, y_j) does not contain any x_i, then put $([y_{j-1}, y_j], t_j)$ in D_1;

if (y_{j-1}, y_j) contains x_i for some i, then put $([y_{j-1}, x_i], t_j)$ and $([x_i, y_j], t_j)$ in D_1.

Although D_1 is not a δ-fine partition of $[a, b]$, we have $S_g(f, D_1) = S_g(f, D)$ because $|y_j - y_{j-1}| < \beta$ and every interval in D_1 is a subset of some interval in D. We rewrite D_1 as $\{([z_{k-1}, z_k], s_k) \mid k = 1, 2, \ldots, N_2\}$ and for each $1 \leq i \leq N$, let $P_i = \{k : [z_{k-1}, z_k] \subseteq [x_{i-1}, x_i]\}$. It is clear that each $k \, (1 \leq k \leq N_2)$ belongs to a unique P_i and $k \in P_i$ implies $|s_k - x_i| < 2\beta < 2\sigma$. Since $\bigcup_{k \in P_i} [z_{k-1}, z_k] = [x_{i-1}, x_i]$, we have

$$\begin{aligned}
||S_g(f, D) - S_g(f, D_0)||_r &= ||S_g(f, D_1) - S_g(f, D_0)||_r \\
&= ||\sum_{i=1}^{N} \sum_{k \in P_i} f(s_k)(g(z_k) - g(z_{k-1})) \\
&\qquad - \sum_{i=1}^{N} f(x_i)(g(x_i) - g(x_{i-1}))||_r \\
&= ||\sum_{i=1}^{N} \sum_{k \in P_i} f(s_k)(g(z_k) - g(z_{k-1})) \\
&\qquad - \sum_{i=1}^{N} \sum_{k \in P_i} f(x_i)(g(z_k) - g(z_{k-1}))||_r \\
&= ||\sum_{i=1}^{N} \sum_{k \in P_i} (f(s_k) - f(x_i))(g(z_k) - g(z_{k-1}))||_r \\
&< \varepsilon \sum_{i=1}^{N} \sum_{k \in P_i} ||(g(z_k) - g(z_{k-1}))||_q \\
&\leq \varepsilon M.
\end{aligned}$$

Let D' and D'' be any two δ-fine partitions of $[a, b]$. Now we have

$$\begin{aligned} ||S_g(f, D') - S_g(f, D'')||_r &\leq ||S_g(f, D') - S_g(f, D_0)||_r + ||S_g(f, D_0) - S_g(f, D'')||_r \\ &< 2M\varepsilon. \end{aligned}$$

Since ε is arbitrary, f is integrable with respect to g on $[a, b]$. $\square$

The following definitions and thorem are important tools in proving the *Monotone Convergence Theorem* discussed in the last section.

Definition 3.19. For each $n \in \mathbb{N}$, let $f_n : [a, b] \to L^p$.
(i) We say that $f_1 \leq f_2$ on $[a, b]$ if and only if $f_1(x) \leq f_2(x)$ for all $x \in [a, b]$.
(ii) A function $g : [a, b] \to L^q$ is said to be *non-decreasing* on $[a, b]$ if and only if $x < y$ implies $g(x) \leq g(y)$ for all $x, y \in [a, b]$.
(iii) A sequence (f_n) is said to be *increasing* if and only if $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$.

Theorem 3.20. *Let (X, M, μ) be a measure space such that $\mu(X) < \infty$. Let $f : [a, b] \to L^p$ be integrable with respect to a non-decreasing function $g : [a, b] \to L^q$ on $[a, b]$. Then $f \geq 0$ implies $\int_a^b f dg \geq 0$.*

Proof. By the construction of the integral $\int_a^b f dg$, there exists a sequence $(S_g(f, D_n))$ in L^r such that $\lim_{n\to\infty} S_g(f, D_n) = \int_a^b f dg$. And for each $n \in \mathbb{N}$,

$$S_g(f, D_n) = \sum_{i=1}^{P_n} f(t_i^n)(g(x_i^n) - g(x_{i-1}^n))$$

where $D_n = \{([x_{i-1}^n, x_i^n], t_i^n) \mid i = 1, 2, \ldots, P_n\}$. Hence $S_g(f, D_n)$ is a non-negative function for all n. By lemma 2.25, we have $\int_a^b f dg \geq 0$. $\square$

Corollary 3.21. *Let (X, M, μ) be a measure space such that $\mu(X) < \infty$. Let $f_1 : [a, b] \to L^p, f_2 : [a, b] \to L^p$ be integrable with respect to a non-decreasing function $g : [a, b] \to L^q$ on $[a, b]$. Then $f_1 \leq f_2$ on $[a, b]$ implies $\int_a^b f_1 dg \leq \int_a^b f_2 dg$.*

Proof. Since $f_1 \leq f_2$ on $[a,b]$, $0 \leq f_2 - f_1$ on $[a,b]$. By theorem 3.20, we get

$$\begin{aligned} 0 &\leq \int_a^b (f_2 - f_1)dg \\ &= \int_a^b f_2 dg - \int_a^b f_1 dg. \end{aligned}$$

Hence, $\int_a^b f_1 dg \leq \int_a^b f_2 dg$. □

3.3 The Saks-Henstock Lemma

In this section, we discuss the important lemma, Saks-Henstock Lemma, that plays a major role in theory of the Henstock integral. The lemma states that good approximations over the entire interval yield good approximations over unions of subintervals.

Theorem 3.22. *Let f be integrable with respect to g on $[a,b]$, and for $\varepsilon > 0$, let δ be a gauge on $[a,b]$ such that for any δ-fine partition D of $[a,b]$*

$$||S_g(f,D) - \int_a^b f dg||_r < \varepsilon.$$

Then for any δ-fine subpartition $D' = \{([x_i,y_i],t_i) \mid i = 1,2,\ldots,n\}$ of $[a,b]$,

$$||\sum_{i=1}^{n}\{f(t_i)[g(y_i) - g(x_i)] - \int_{x_i}^{y_i} f dg\}||_r = ||S_g(f,D') - \int_{\bigcup_{i=1}^{n}[x_i,y_i]} f dg||_r < \varepsilon.$$

Proof. Let $\{K_j : 1 \leq j \leq m\}$ be the collection of closed intervals in $[a,b]$ such that the union of it and $\{[x_i,y_i] \mid i = 1,2,\ldots,n\}$ is $[a,b]$. Let $\eta > 0$ and for each j, let δ_j be a gauge on K_j such that if D_j is a δ_j-fine partition of K_j, then

$$||S_g(f,D_j) - \int_{K_j} f dg||_r < \frac{\eta}{m}. \tag{3.9}$$

Clearly we may assume that $\delta_j(t) \leq \delta(t)$ for all $t \in K_j$. Let $D = D' \cup \bigcup_{j=1}^{m} D_j$. Then D is a δ-fine partition of $[a,b]$ and

$$||S_g(f,D) - \int_a^b f dg||_r < \varepsilon. \tag{3.10}$$

Combining the inequalities (3.9) and (3.10), we obtain

$$\begin{aligned}
||S_g(f, D') - \int_{\bigcup_{i=1}^n [x_i, y_i]} fdg||_r &= ||S_g(f, D') + \sum_{j=1}^m S_g(f, D_j) - \int_{\bigcup_{i=1}^n [x_i, y_i]} fdg \\
&\quad - \sum_{j=1}^m \int_{K_j} fdg + \sum_{j=1}^m [S_g(f, D_j) - \int_{K_j} fdg]||_r \\
&\le ||S_g(f, D) - \int_a^b fdg||_r + \sum_{j=1}^m ||S_g(f, D_j) - \int_{K_j} fdg||_r \\
&< \varepsilon + \eta.
\end{aligned}$$

Since $\eta > 0$ is arbitrary, then the proof is complete. □

Corollary 3.23. *Let f be integrable with respect to g on $[a, b]$ and for $\varepsilon > 0$ let δ be a gauge on $[a, b]$ such that for any δ-fine partition $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ of $[a, b]$*

$$||S_g(f, D) - \int_a^b fdg||_r < \varepsilon.$$

Then for each $i = 1, 2, \ldots, n$,

$$||f(t_i)[g(x_i) - g(x_{i-1})] - \int_{x_{i-1}}^{x_i} fdg||_r < \varepsilon.$$

3.4 The Fundamental Theorem of Calculus

In this section we see an important application of the Saks-Henstock Lemma; The Fundamental Theorem of Calculus.

Consider a function $f : [a, b] \to L^p$ which is integrable with respect to a function $g : [a, b] \to L^q$ on $[a, b]$. For any $x \in [a, b]$, the integral of f on $[a, x]$ exists, so we can define a function $F : [a, b] \to L^r$ by

$$F(x) = \int_a^x fdg,$$

for $x \in [a, b]$.

Theorem 3.24. *If $f : [a,b] \to L^p$ is integrable with respect to a continuous function $g : [a,b] \to L^q$ on $[a,b]$, then the integral function*

$$F(x) = \int_a^x f dg \qquad \text{for all } x \in [a,b],$$

is continuous on $[a,b]$.

Proof. Let $c \in [a,b]$ and let $\varepsilon > 0$. Since g is continuous at c, there is an $\alpha > 0$ such that for any $x \in [a,b]$ with $|x-c| < \alpha$,

$$||g(x) - g(c)||_q < \frac{\varepsilon}{||f(c)||_p + 1}. \tag{3.11}$$

Let δ be a gauge on $[a,b]$ such that for any δ-fine partition D of $[a,b]$,

$$||S_g(f,D) - \int_a^b f dg||_r < \varepsilon.$$

Let $\eta = \min\{\delta(c), \alpha\}$. For each $x \in [a,b]$ with $|x-c| < \eta$, the interval with endpoints x, c and tag c is δ-fine subpartition of $[a,b]$. Apply the Saks-Henstock Lemma and the inequality (3.11),

$$\begin{aligned} ||F(x) - F(c)||_r &\leq ||F(x) - F(c) - f(c)[g(x) - g(c)]||_r + ||f(c)[g(x) - g(c)]||_r \\ &< 2\varepsilon. \end{aligned}$$

Therefore, the function F is continuous at c. □

Definition 3.25. A non-negative real-valued function η defined on $[a,b]$ is said to be *a countably closed gauge on* $[a,b]$ if the set of its zeros is countable.

The next theorem, we give an interesting result that allows a positive function (gauge) to be a non-negative function for which the set of its zeros is countable. The idea used to the prove theorem follows from the work of Rudolf Výborný [10].

Theorem 3.26. *Let $g : [a,b] \to L^q$ be a continuous function. A function $f : [a,b] \to L^p$ is integrable with respect to g on $[a,b]$ if and only if there exists a* *continuous function $F : [a, b] \to L^r$ such that for every $\varepsilon > 0$ there is a countably closed gauge η with the property that*

$$||\sum_{i=1}^{n}\{f(t_i)[g(y_i) - g(x_i)] - [F(y_i) - F(x_i)]\}||_r < \varepsilon, \tag{3.12}$$

for any η-fine subpartition $\{([x_i, y_i], t_i) \mid i = 1, \ldots, n\}$ of $[a, b]$. If the condition is satisfied then $\int_a^b f dg = F(b) - F(a)$.

Proof. Suppose that f is integrable. Let $F : [a, b] \to L^r$ be the indefinite integral of f. Since g is continuous on $[a, b]$, so is F. Then directly from the Saks-Henstock lemma, the inequality (3.12) is satisfied.

Now, we assume the converse. We will show that f is integrable with respect to g on $[a, b]$. Let $\varepsilon > 0$. There exists a countably closed gauge η satisfying the property (3.12). Let $r_1, r_2, \ldots, r_n, \ldots$ be the enumeration of the zeros of η such that for $i < j$, $r_i < r_j$ and let $\delta'(r_n) > 0$ be such that

$$||F(u) - F(v)||_r < \frac{\varepsilon}{2^{n+2}} \quad \text{and} \quad ||f(r_n)[g(u) - g(v)]||_r < \frac{\varepsilon}{2^{n+2}} \tag{3.13}$$

for $|u - r_n| < \delta'(r_n)$ and $|v - r_n| < \delta'(r_n)$. Define $\delta : [a, b] \to \mathbb{R}^+$ by

$$\delta(x) = \begin{cases} \delta'(x) & \text{if } x = r_n \\ \eta(x) & \text{if } x \neq r_n. \end{cases}$$

For any δ-fine partition $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ denote by

$$\sum{}'\{f(t_i)[g(x_i) - g(x_{i-1})] - [F(x_i) - F(x_{i-1})]\} \text{ (or shortly by } \sum{}'\text{)}$$

the sum is taken in which $t_i \neq r_i$ for all i and by

$$\sum{}''\{f(t_i)[g(x_i) - g(x_{i-1})] - [F(x_i) - F(x_{i-1})]\} \text{ (or shortly by } \sum{}''\text{)}$$

for the remaining indices i.

It follows from the inequalities (3.13) that

$$||\sum{}''||_r \leq \sum ||\{f(t_i)[g(x_i)-g(x_{i-1})]-[F(x_i)-F(x_{i-1})]\}||_r$$
$$< \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+2}} + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+2}} = \frac{\varepsilon}{2}.$$

Moreover, by assumption we have $||\sum{}'||_r < \frac{\varepsilon}{2}$.

Let $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ be a δ-fine partition. We have

$$||S_g(f,D) - [F(b)-F(a)]||_r = ||\sum{}' + \sum{}''||_r$$
$$\leq ||\sum{}'||_r + ||\sum{}''||_r < \varepsilon.$$

Since ε is arbitrary, f is integrable. □

In order to proceed the Fundamental Theorem of Calculus, we use the idea of the Straddle Lemma (see in chapter 2, Lemma 2.1) to define a derivative of a function F with respect to a function g.

Definition 3.27. Let $F : [a,b] \to L^r$ and $g : [a,b] \to L^q$. Then F is said to be *differentiable with respect to* g on $[a,b]$ if there exists a function $f : [a,b] \to L^p$ satisfying the following condition: for any $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that for any $t \in [a,b]$, if $u, v \in [a,b] \cap (t-\delta(t), t+\delta(t))$, then

$$||F(v) - F(u) - f(t)[g(v)-g(u)]||_r \leq \varepsilon ||g(v)-g(u)||_q.$$

The function f is called the *derivative* (with respect to g) of F on $[a,b]$.

From the definition of derivative, it is obvious that the zero function is the derivative of constant functions.

Theorem 3.28. *If $F : [a,b] \to L^r$ is differentiable with respect to a continuous function $g : [a,b] \to L^q$ on $[a,b]$, then F is continuous on $[a,b]$.*

Proof. Let $c \in [a,b]$ and let $\varepsilon > 0$. There exists a positive number α such that for any $t \in [a,b]$ with $|t-c| < \alpha$, we have

$$||g(t) - g(c)||_q < \frac{\varepsilon}{||f(c)||_p + 1}. \tag{3.14}$$

Let η be a gauge on $[a,b]$ such that

for any $t \in [a,b]$, if $u, v \in [a,b] \cap (t-\eta(t), t+\eta(t))$, then

$$||F(v) - F(u) - f(t)[g(v) - g(u)]||_r \leq ||g(v) - g(u)||_q. \tag{3.15}$$

Let $\delta = \min\{\alpha, \eta(c)\}$. Suppose $h \in [a,b]$ such that $|h-c| < \delta$. By the inequalities (3.14) and (3.15), we get

$$\begin{aligned} ||F(h) - F(c)||_r &< (||f(c)||_p + 1)\, ||g(h) - g(c)||_q \\ &< (||f(c)||_p + 1) \frac{\varepsilon}{||f(c)||_p + 1} = \varepsilon. \end{aligned}$$

We complete the proof. □

Theorem 3.29. *(The Fundamental Theorem of Calculus) Let $g : [a,b] \to L^q$ be of bounded variation on $[a,b]$. If $f : [a,b] \to L^p$ is the derivative with respect to g of $F : [a,b] \to L^r$ on $[a,b]$, then f is integrable with respect to g on $[a,b]$ and*

$$\int_a^b f dg = F(b) - F(a).$$

Proof. Let M be a positive real number larger than the variation of g on $[a,b]$. Let $\varepsilon > 0$. Since f is the derivative with respect to g of F, there is a gauge δ on $[a,b]$ such that for any δ-fine partition $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ of $[a,b]$, we have

$$||F(x_i) - F(x_{i-1}) - f(t_i)[g(x_i) - g(x_{i-1})]||_r \leq \frac{\varepsilon}{M} ||g(x_i) - g(x_{i-1})||_q. \tag{3.16}$$

Let $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ be a δ-fine partition of $[a,b]$. From the inequality (3.16) and the telescoping sum on $F(b) - F(a)$, we have

$$
\begin{aligned}
||F(b) - F(a) - S_g(f, D)||_r &= ||\sum_{i=1}^{n}\{[F(x_i) - F(x_{i-1})] - f(t_i)[g(x_i) - g(x_{i-1})]\}||_r \\
&\leq \sum_{i=1}^{n} ||F(x_i) - F(x_{i-1}) - f(t_i)[g(x_i) - g(x_{i-1})]\}||_r \\
&\leq \frac{\varepsilon}{M} \sum_{i=1}^{n} ||g(x_i) - g(x_{i-1})||_q \\
&< \ \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, we conclude that f is integrable with $\int_a^b f dg = F(b) - F(a)$.

□

A consequence of the Fundamental Theorem of Calculus is the integration by parts. Since $\frac{1}{p} + \frac{1}{r} = \frac{1}{(\frac{pr}{p+r})}$ and $\frac{1}{(\frac{pr}{p+r})} + \frac{1}{q} = \frac{1}{(\frac{r}{2})}$, the integral $\int_a^b (f_1 F_2 + F_1 f_2) dg$ considered in the next theorem can be defined when $r \geq 2$.

Theorem 3.30. *(Integration by Parts) Let $r \geq 2$ and let $g : [a, b] \to L^q$ be of bounded variation on $[a, b]$. Suppose that $f_1 : [a, b] \to L^p$ and $f_2 : [a, b] \to L^p$ are the derivative with respect to g on $[a, b]$ of functions F_1 and F_2, respectively. Then $\int_a^b (f_1 F_2 + F_1 f_2) dg$ exists and*

$$\int_a^b (f_1 F_2 + F_1 f_2) dg = F_1(b) F_2(b) - F_1(a) F_2(a).$$

Proof. Let $M \in \mathbb{R}^+$ be such that $||F_1(x)||_r \leq M$ and $||F_2(x)||_r \leq M$ for all $x \in [a, b]$, and also larger than the variation of g on $[a, b]$. Since F_1 and F_2 are continuous and differentiable, for each $t \in [a, b]$, given $\varepsilon > 0$, let $\sigma_t > 0$ such that

$$||f_1(t)||_p \cdot ||F_2(t) - F_2(x)||_r < \frac{\varepsilon}{4M} \tag{3.17}$$

and

$$||f_2(t)||_p \cdot ||F_1(t) - F_1(x)||_r < \frac{\varepsilon}{4M} \tag{3.18}$$

for any $x \in [a,b]$ for which $|x-t| < \sigma_t$ and we choose a positive function α on $[a,b]$ such that for any α-fine partition $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ of $[a,b]$,

$$||[F_1(x_i) - F_1(x_{i-1})] - f_1(t_i)[g(x_i) - g(x_{i-1})]||_r \leq \frac{\varepsilon}{4M^2}||g(x_i) - g(x_{i-1})||_q, \quad (3.19)$$

$$||[F_2(x_i) - F_2(x_{i-1})] - f_2(t_i)[g(x_i) - g(x_{i-1})]||_r \leq \frac{\varepsilon}{4M^2}||g(x_i) - g(x_{i-1})||_q. \quad (3.20)$$

Let $\delta : [a,b] \to \mathbb{R}^+$ defined by $\delta(t) = \min\{\alpha(t), \sigma_t\}$ for all $t \in [a,b]$.

Let $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ be a δ-fine partition of $[a,b]$, we have

$$\begin{aligned}
&||\sum_{i=1}^{n}[f_1(t_i)F_2(t_i) + f_2(t_i)F_1(t_i)][g(x_i) - g(x_{i-1})] - [F_1(b)F_2(b) - F_1(a)F_2(a)]||_{\frac{r}{2}} \\
&\quad = ||\sum_{i=1}^{n}\{f_1(t_i)F_2(t_i)[g(x_i) - g(x_{i-1})] + f_2(t_i)F_1(t_i)][g(x_i) - g(x_{i-1})] \\
&\qquad - [F_1(x_i)F_2(x_i) - F_1(x_{i-1})F_2(x_{i-1})]\}||_{\frac{r}{2}} \\
&\quad \leq \sum_{i=1}^{n}||f_1(t_i)F_2(t_i)[g(x_i) - g(x_{i-1})] - F_2(x_i)[F_1(x_i) - F_1(x_{i-1})]||_{\frac{r}{2}} \\
&\qquad + \sum_{i=1}^{n}||f_2(t_i)F_1(t_i)[g(x_i) - g(x_{i-1})] - F_1(x_i)[F_2(x_i) - F_2(x_{i-1})]||_{\frac{r}{2}} \\
&\quad \leq \sum_{i=1}^{n}||f_1(t_i)||_p||F_2(t_i) - F_2(x_i)||_r||g(x_i) - g(x_{i-1})||_q \\
&\qquad + \sum_{i=1}^{n}||F_2(x_i)||_r||f_1(t_i)[g(x_i) - g(x_{i-1})] - [F_1(x_i) - F_1(x_{i-1})]||_r \\
&\qquad + \sum_{i=1}^{n}||f_2(t_i)||_p||F_1(t_i) - F_1(x_i)||_r||g(x_i) - g(x_{i-1})||_q \\
&\qquad + \sum_{i=1}^{n}||F_1(x_i)||_r||f_2(t_i)[g(x_i) - g(x_{i-1})] - [F_2(x_i) - F_2(x_{i-1})]||_r \\
&\quad < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \; = \; \varepsilon,
\end{aligned}$$

by the inequalities (3.17), (3.18), (3.19) and (3.20). Therefore $\int_a^b (f_1F_2 + F_1f_2)dg$ exists and $\int_a^b (f_1F_2 + F_1f_2)dg \; = \; F_1(b)F_2(b) - F_1(a)F_2(a)$. $\square$

3.5 Convergence Theorems

In this section we will show that the major convergence theorems hold for our integral.

Theorem 3.31. *(Uniform Convergence Theorem) Let $g : [a,b] \to L^q$ be of bounded variation on $[a,b]$ and for each $n \in \mathbb{N}$, $f_n : [a,b] \to L^p$. If (f_n) is a sequence of integrable functions with respect to g on $[a,b]$ and converges to $f : [a,b] \to L^p$ uniformly on $[a,b]$, then f is integrable with respect to g on $[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg\,[L^r].$$

Proof. Let M be the variation of g over $[a,b]$. We will first show that $(\int_a^b f_n dg)$ is a Cauchy sequence. Let $\varepsilon > 0$. Since the sequence (f_n) converges to f uniformly on $[a,b]$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $||f_n(t) - f(t)||_p < \varepsilon$ for all $t \in [a,b]$. Consequently, for any $m, n \geq N$ we have $||f_n(t) - f_m(t)||_p < \varepsilon$ for all $t \in [a,b]$. Let $m, n \geq N$. Let δ_m and δ_n be gauges on $[a,b]$ such that

$$||S_g(f_m, D') - \int_a^b f_m dg||_r < \varepsilon \quad \text{and} \quad ||S_g(f_n, D'') - \int_a^b f_n dg||_r < \varepsilon, \tag{3.21}$$

for any δ_m-fine partition D' of $[a,b]$ and any δ_n-fine partition D'' of $[a,b]$, respectively.

Let δ be a gauge on $[a,b]$ defined by $\delta(t) = \min\{\delta_m(t), \delta_n(t)\}$ for $t \in [a,b]$. By Cousin's lemma, there exists a δ-fine partition $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ of $[a,b]$ and hence D is a δ_m-fine and δ_n-fine partition of $[a,b]$. Moreover, we have

$$\begin{aligned} ||S_g(f_m, D) - S_g(f_n, D)||_r &\leq \sum_{i=1}^{n} ||f_m(t_i) - f_n(t_i)||_p ||g(x_i) - g(x_{i-1})||_q \\ &< \varepsilon \sum_{i=1}^{n} ||g(x_i) - g(x_{i-1})||_q \leq M\varepsilon. \end{aligned} \tag{3.22}$$

Applying the triangle inequality to the inequalities (3.21) and (3.22), we obtain

$$||\int_a^b f_m dg - \int_a^b f_n dg||_r < (M+2)\varepsilon.$$

Therefore the sequence $(\int_a^b f_n dg)$ is a Cauchy sequence and so converges in L^r to some $A \in L^r$. We now show that f is integrable with respect to g on $[a,b]$ with the integral A. If $\bar{D} = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ is any tagged partition of $[a,b]$ and $k \geq N$, then

$$\begin{aligned} ||S_g(f_k, \bar{D}) - S_g(f, \bar{D})||_r &\leq \sum_{i=1}^{n} ||f_k(t_i) - f(t_i)||_p ||g(x_i) - g(x_{i-1})||_q \\ &< M\varepsilon. \end{aligned} \tag{3.23}$$

Since $(\int_a^b f_n dg)$ converges to A in L^r, there exists $N' \in \mathbb{N}$ such that for any $n \geq N'$,

$$||\int_a^b f_n dg - A||_r < \varepsilon. \tag{3.24}$$

Now choose a fixed number $k \geq \max\{N, N'\}$ and let δ_k be a gauge on $[a,b]$ such that for any δ_k-fine partition D' of $[a,b]$,

$$||S_g(f_k, D') - \int_a^b f_k dg||_r < \varepsilon \tag{3.25}$$

Let D' is a δ_k-fine partition of $[a,b]$. Applying the triangle inequality again to the inequalities (3.23), (3.24) and (3.25), we have

$$||S_g(f, D') - A||_r < (M+2)\varepsilon.$$

Since ε is arbitrary, f is integrable with respect to g on $[a,b]$ and $\int_a^b f dg = A$. $\square$

Corollary 3.32. *Let $g : [a,b] \to L^q$ be of bounded variation on $[a,b]$ and for each $n \in \mathbb{N}$, $f_n : [a,b] \to L^p$ be continuous function. If (f_n) converges to f uniformly on $[a,b]$, then f is integrable with respect to g on $[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg \, [L^r].$$

We now give another situation concerning the convergence theorems that for a given $\varepsilon > 0$, there is a gauge that satisfies the definition of integrability for every function in the sequence.

Definition 3.33. Let $g : [a,b] \to L^q$ and for each $n \in \mathbb{N}$, $f_n : [a,b] \to L^p$. Let (f_n) be a sequence of integrable with respect to g on $[a,b]$. The sequence (f_n) is *equi-integrable with respect to* g on $[a,b]$ if for each $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any δ-fine partition D of $[a,b]$,

$$||S_g(f_n, D) - \textstyle\int_a^b f_n dg||_r < \varepsilon \text{ for all } n \in \mathbb{N}.$$

We will show that equi-integrability and pointwise convergence of the sequence of functions imply that the limit function is integrable.

Theorem 3.34. *Let $g : [a,b] \to L^q$ be of bounded variation on $[a,b]$ and for each $n \in \mathbb{N}$, $f_n : [a,b] \to L^p$. If the sequence (f_n) is equi-integrable with respect to g on $[a,b]$ and the sequence converges pointwise to $f : [a,b] \to L^p$ on $[a,b]$, then f is integrable with respect to g on $[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg \, [L^r].$$

Proof. Let M be the variation of g over $[a,b]$. Firstly, we will show that $(\int_a^b f_n dg)$ is a Cauchy sequence in L^r. Let $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that for any δ-fine partition D of $[a,b]$,

$$||S_g(f_n, D) - \textstyle\int_a^b f_n dg||_r < \frac{\varepsilon}{3} \text{ for all } n \in \mathbb{N}.$$

Let $D_0 = \{([x'_{i-1}, x'_i], t'_i) \mid i = 1, 2, \ldots, s\}$ be a δ-fine partition of $[a,b]$. Then for all $n \in \mathbb{N}$,

$$||S_g(f_n, D_0) - \int_a^b f_n dg||_r < \frac{\varepsilon}{3}. \tag{3.26}$$

Since (f_n) converges pointwise to f on $[a,b]$, there exists $N \in \mathbb{N}$ such that

$$||f_m(t_i') - f_n(t_i')||_p < \tfrac{\varepsilon}{3M} \text{ for all } i = 1, 2, \ldots, s \text{ and } m, n \geq N$$

and hence

$$\begin{aligned} ||S_g(f_m, D_0) - S_g(f_n, D_0)||_r &\leq \sum_{i=1}^{s} ||f_m(t_i') - f_n(t_i')||_p ||g(x_i') - g(x_{i-1}')||_q \\ &< \frac{\varepsilon}{3} \end{aligned} \tag{3.27}$$

Applying the triangle inequality to (3.26) and (3.27), we get

$$|| \textstyle\int_a^b f_m dg - \int_a^b f_n dg||_r < \varepsilon \text{ for all } m, n \geq N.$$

Since ε is arbitrary, $(\int_a^b f_n dg)$ is a Cauchy sequence in L^r and hence $(\int_a^b f_n dg)$ converges to a unique A in L^r. Let D be any δ-fine partition of $[a, b]$ and $k \in \mathbb{N}$ such that

$$|| \int_a^b f_k dg - A||_r < \frac{\varepsilon}{3} \quad \text{and} \quad ||S_g(f_k, D) - S_g(f, D)||_r < \frac{\varepsilon}{3} \tag{3.28}$$

By the inequality (3.28), the equi-integrability of (f_n) and the triangle inequality, we have

$$||S_g(f, D) - A||_r < \varepsilon.$$

This show that f is integrable with respect to g with integral A. □

The next theorem is inspired by the work of R.A. Gordon [6].

Theorem 3.35. *Let $g : [a, b] \to L^q$ be of bounded variation on $[a, b]$ and for each $n \in \mathbb{N}$, $f_n : [a, b] \to L^p$. Suppose (f_n) is a sequence of integrable with respect to g on $[a, b]$ and converges pointwise to $f : [a, b] \to L^p$ on $[a, b]$. Then f is integrable with respect to g on $[a, b]$ and $\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg\, [L^r]$ if and only if for each $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that for any δ-fine partition D, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then*

$$||S_g(f_n, D) - \int_a^b f_n dg||_r < \varepsilon.$$

Proof. Let M be the variation of g over $[a,b]$. Suppose f is integrable with respect to g on $[a,b]$ and $\lim\limits_{n\to\infty}\int_a^b f_n dg = \int_a^b f dg\,[L^r]$. Let $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$||\int_a^b f dg - \int_a^b f_n dg||_r < \frac{\varepsilon}{3} \tag{3.29}$$

and there exists a gauge δ on $[a,b]$ such that for any δ-fine partition D of $[a,b]$,

$$||S_g(f,D) - \int_a^b f dg||_r < \frac{\varepsilon}{3}. \tag{3.30}$$

Let $D = \{([x_{i-1},x_i],t_i) \mid i = 1,2,\ldots,s\}$ be a δ-fine partition of $[a,b]$. Since the sequence (f_n) converges pointwise to f on $[a,b]$, we choose $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $||f_n(t_i) - f(t_i)||_p < \frac{\varepsilon}{3M}$ for all $i = 1,2,\ldots,s$ and hence

$$||S_g(f_n,D) - S_g(f,D)||_r < \frac{\varepsilon}{3}. \tag{3.31}$$

Now, let $n \geq \max\{N_1,N_2\}$. Applying the triangle inequality to (3.29), (3.30) and (3.31), we get

$$||S_g(f_n,D) - \int_a^b f_n dg||_r < \varepsilon.$$

Now, assume the converse. We will first show that $(\int_a^b f_n dg)$ is a Cauchy sequence. Let $\varepsilon > 0$. There exists a gauge δ on $[a,b]$ satisfying the assumption and we choose $D_0 = \{([x'_{i-1},x'_i],t'_i) \mid i = 1,2,\ldots,s\}$ to be a δ-fine partition of $[a,b]$. By the assumption, there exists $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then

$$||S_g(f_n,D_0) - \int_a^b f_n dg||_r < \frac{\varepsilon}{3}. \tag{3.32}$$

Since (f_n) converges pointwise on $[a,b]$, there exists $N_1 \in \mathbb{N}$ such that if $m,n \geq N_1$, then $||f_m(t'_i) - f_n(t'_i)||_p < \frac{\varepsilon}{3M}$ for all $i = 1,2,\ldots,s$, and hence,

$$||S_g(f_m,D) - S_g(f_n,D)||_r < \frac{\varepsilon}{3}. \tag{3.33}$$

Suppose $m,n \geq \max\{N_0,N_1\}$. Applying the triangle inequality again to (3.32) and (3.33), we have $||\int_a^b f_m dg - \int_a^b f_n dg||_r < \varepsilon$. It follows that $(\int_a^b f_n dg)$ is a

Cauchy sequence in L^r. Let $A \in L^r$ be the limit of the sequence. We claim that $\int_a^b f dg = A$. Let $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, k\}$ be any δ-fine partition of $[a, b]$. By the hypothesis, there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then

$$||S_g(f_n, D) - \int_a^b f_n dg||_r < \frac{\varepsilon}{3}. \tag{3.34}$$

Since the sequence $(\int_a^b f_n dg)$ converges to A in L^r, there exists $N_3 \in \mathbb{N}$ such that if $n \geq N_3$, then

$$||\int_a^b f_n dg - A||_r < \frac{\varepsilon}{3}. \tag{3.35}$$

Since the sequence (f_n) converges pointwise to f, there exists N_4 such that if $n \geq N_4$, then

$$||S_g(f, D) - S_g(f_n, D)||_r < \frac{\varepsilon}{3}. \tag{3.36}$$

Let $k = \max\{N_2, N_3, N_4]$. Now by combining (3.34), (3.35) and (3.36), we have

$$||S_g(f, D) - A||_r < \varepsilon.$$

Hence f is integrable with respect to g on $[a, b]$ and $\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg \, [L^r]$. $\square$

We close this chapter by proving a major convergence theorem called the Monotone Convergence Theorem. It requires pointwise (rather than uniform) convergence and monotonicity of the sequence of functions. We will discuss only the case of increasing sequence of functions.

Theorem 3.36. *(Monotone Convergence Theorem) Let (X, M, μ) be a measure space such that $\mu(X) < \infty$. Let $g : [a, b] \to L^q$ be of bounded variation and non-decreasing on $[a, b]$. For each $n \in \mathbb{N}$, let $f_n : [a, b] \to L^p$ be integrable with respect to g on $[a, b]$ and the sequence (f_n) increasing and converges pointwise to $f : [a, b] \to L^p$ on $[a, b]$. If the sequence $(\int_a^b f_n dg)$ converges uniformly to some $A \in L^r$ on X, then f is integrable with respect to g on $[a, b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg \, [L^r].$$

Proof. Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that if $n \geq N$ then for all $y \in X$,

$$|(\int_a^b f_n dg)(y) - A(y)| < \frac{\varepsilon}{\mu(X)^{1/r}}. \tag{3.37}$$

Thus for all $n \geq N$,

$$||\int_a^b f_n dg - A||_r < \varepsilon. \tag{3.38}$$

Let $N_1 \in \mathbb{N}$ such that $N_1 \geq N$ and $\frac{1}{2^{N_1-2}} < \varepsilon$. Since f_k is integrable, for each k there exists a gauge δ_k on $[a,b]$ such that if D is a δ_k-fine partition of $[a,b]$, then

$$||S_g(f_k, D) - \int_a^b f_k dg||_r < \frac{1}{2^k}. \tag{3.39}$$

Since (f_n) converges pointwise to f on $[a,b]$, for each $x \in [a,b]$ there is $k(x) \geq N_1$ such that

$$||f(x) - f_{k(x)}(x)||_r < \frac{\varepsilon}{M} \tag{3.40}$$

where $M = Var(g, [a,b])$. Now define $\delta(t) = \delta_{k(t)}(t)$ for all $t \in [a,b]$. We will show that f is integrable with integral A. Let $D = \{([x_{i-1}, x_i], t_i) \mid i = 1, 2, \ldots, n\}$ be a δ-fine partition of $[a,b]$. We consider the following inequality,

$$\begin{aligned} ||S_g(f,D) - A||_r \leq {} & ||\sum_{i=1}^n f(t_i)[g(x_i) - g(x_{i-1})] - \sum_{i=1}^n f_{k(t_i)}(t_i)[g(x_i) - g(x_{i-1})]||_r \\ & + ||\sum_{i=1}^n f_{k(t_i)}(t_i)[g(x_i) - g(x_{i-1})] - \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f_{k(t_i)} dg||_r \qquad (3.41) \\ & + ||\sum_{i=1}^n \int_{x_{i-1}}^{x_i} f_{k(t_i)} dg - A||_r. \end{aligned}$$

By the inequality (3.40), the first term of the right handside of the above inequality is dominated by ε, since

$$\sum_{i=1}^n ||f(t_i) - f_{k(t_i)}(t_i)||_p ||[g(x_i) - g(x_{i-1})]||_q < \varepsilon.$$

To estimate the second term, let $N_2 = \max\{k(t_1), k(t_2), \ldots, k(t_n)\} \geq N_1$. We note that the finite sum of the second term can be written as an iterated sum:

first over all values of i such that $k(t_i) = s$ for some natural number $s \geq N_1$, and then over $s \in \{N_1, \ldots, N_2\}$. The Saks-Henstock Lemma implies that

$$||\sum_{k(t_i)=s} \{f_{k(t_i)}(t_i)[g(x_i) - g(x_{i-1})] - \int_{x_{i-1}}^{x_i} f_{k(t_i)} dg\}||_r < \frac{1}{2^{s-1}}. \tag{3.42}$$

If we sum the inequality (3.42) over $s \in \{N_1, \ldots, N_2\}$. We find that the second term in the inequality (3.41) is dominated by ε, since

$$\sum_{s=N_1}^{N_2} \frac{1}{2^{s-1}} < \sum_{s=N_1}^{\infty} \frac{1}{2^{s-1}} = \frac{1}{2^{N_1-2}} < \varepsilon.$$

We now estimate the third term in (3.41). Since the sequence (f_n) is increasing and $N_1 \leq k(t_i) \leq N_2$ for all i, $f_{N_1} \leq f_{k(t_i)} \leq f_{N_2}$ and hence

$$\int_{x_{i-1}}^{x_i} f_{N_1} dg \leq \int_{x_{i-1}}^{x_i} f_{k(t_i)} dg \leq \int_{x_{i-1}}^{x_i} f_{N_2} dg.$$

Summing these inequalities for $i = 1, 2, \ldots, n$, we obtain

$$\int_a^b f_{N_1} dg \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f_{k(t_i)} dg \leq \int_a^b f_{N_2} dg.$$

Thus there is $E \subseteq X$ such that $\mu(X \backslash E) = 0$ and for all $y \in E$,

$$(\int_a^b f_{N_1} dg)(y) \leq (\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f_{k(t_i)} dg)(y) \leq (\int_a^b f_{N_2} dg)(y).$$

By the inequality (3.37), we have

$$|(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f_{k(t_i)} dg)(y) - A(y)| < \frac{\varepsilon}{\mu(X)^{1/r}} \text{ for all } y \in E.$$

Therefore

$$\begin{aligned} ||\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f_{k(t_i)} dg - A||_r &= (\int_X |(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f_{k(t_i)} dg)(y) - A(y)|^r d\mu)^{\frac{1}{r}} \\ &= (\int_E |(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f_{k(t_i)} dg)(y) - A(y)|^r d\mu)^{\frac{1}{r}} \\ &< \varepsilon. \end{aligned}$$

Combining the three estimations, we conclude that f is integrable with respect to g on $[a, b]$ and $\lim_{n\to\infty} \int_a^b f_n dg = \int_a^b f dg \, [L^r]$. $\square$

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VITA

Name	Mr. Nopporn Thamrongrat
Date of Birth	28 March 1986
Place of Birth	Nakhon Si Thammarat, Thailand
Education	B.Sc. (Mathematics, First Class Honors), Prince of Songkhla University, 2008
Scholarship	The Development and Promotion of Science and Technology Talents Projects (DPST)