CHAPTER 5

Harmonic-Oscillator Propagator

In this chapter we present our techniques for calculating the harmonic-oscillator propagator. In the first section we discuss the main ideas of our techniques. In sec. 5.2 we calculate the harmonic-oscillator propagator and show how the prefactor and the exponent of the classical action can be obtained simultaneously. In the last section the conclusions and discussions are given.

5.1 Preliminary

Our technique is a combination of both Devies' and Feynman's methods. We represent the paths as a cosine series and transform the path integral to be the multiple integrals of the coefficients of the series. We restrict ourselves to the discrete-time assumption which Feynman and Devies had not taken into consideration. After performing the integrations and combining all factors together, the required results are obtained. In the following section we calculate the harmonic-oscillator propagator by using these ideas. In the next chapter we apply our techniques to the calculation of the non-local harmonic oscillator propagator.

5.2 Calculating the Harmonic-Oscillator Propagator

In order to obtain the prefactor and the exponent of the classical action simultaneously, we express the propagator in the following form :

$$K(x_b,T;x_a,0) = \lim_{N\to\infty}\left(\frac{1}{A}\right)^N \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_0\ldots dx_N \exp\left\{-\frac{m}{2i\hbar\epsilon}\left[\sum_{j=1}^{N}(x_j-x_{j-1})^2\right.\right.$$

$$\left.\left.-\omega^2\epsilon^2\sum_{j=0}^{N}x_j^2\right]\right\}\delta(x_0-x_a)\delta(x_N-x_b) \qquad (5.1)$$

Since all paths originate at the point x_a and terminate at the point x_b, it is convenient to represent them as a cosine series :

$$x(t) = \frac{1}{\sqrt{T}}a_0 + \sqrt{\frac{2}{T}}\sum_{n=1}^{N-1}a_n\cos\frac{n\pi t}{T} + \frac{1}{\sqrt{T}}a_N\cos\frac{N\pi t}{T} \qquad (5.2)$$

where $\frac{1}{\sqrt{T}}$ and $\sqrt{\frac{2}{T}}$ are the chosen weighting factors.

Since we put the boundary points into the path integral, the weighting factors of a_0 and a_N must be properly chosen.

On the assumption of dividing the time T into N discrete steps of length ϵ , eq. (5.2) can be written as

$$x_j = \frac{1}{\sqrt{N\epsilon}}a_0 + \sqrt{\frac{2}{N\epsilon}}\sum_{n=1}^{N-1}a_n\cos\frac{n\pi j}{N} + \frac{(-1)^j}{\sqrt{N\epsilon}}a_N \qquad (5.3)$$

The kinetic-energy terms in the action function of eq. (5.1) can be transformed on to be (the derivations are presented in Appendix A)

$$\sum_{j=1}^{N}(x_j-x_{j-1})^2 = \sum_{n=0}^{N}\left(\frac{4}{\epsilon}\sin^2\frac{n\pi}{2N}\right)a_n^2 \qquad (5.4)$$

Similary, for the potential-energy terms can be transformed to be

$$\sum_{j=0}^{N} x_j^2 = \sum_{m=0}^{N} \frac{1}{\epsilon} a_m^2 \tag{5.5}$$

The path integral in eq. (5.1) now becomes

$$K(x_b,T;x_a,0) = \lim_{N\to\infty} J\left(\frac{1}{A}\right)^N \int_{-\infty}^{\infty}\!\!\cdots\!\int_{-\infty}^{\infty} da_0\cdots da_N \exp\left\{-\frac{m}{2i\hbar\epsilon^2}\sum_{m=0}^{N}\left(4\sin^2\frac{m\pi}{2N} - \omega^2\epsilon^2\right)a_m^2\right\} \delta\left(\frac{1}{\sqrt{T}}\left[a_0+\sqrt{2}\sum_{m=1}^{N-1}a_m + a_N - \sqrt{T}x_a\right]\right)$$

$$\times\,\delta\left(\frac{1}{\sqrt{T}}\left[a_0+\sqrt{2}\sum_{m=1}^{N-1}(-1)^m a_m + (-1)^N a_N - \sqrt{T}x_b\right]\right) \tag{5.6}$$

Using the properties of Dirac delta function,

$$\delta(ax) = \frac{1}{a}\delta(x) \tag{5.7}$$

and

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp\{ik(x-a)\} \tag{5.8}$$

eq. (5.6) can be rewritten as

$$K(x_b,T;x_a,0) = \lim_{N\to\infty}\left(\frac{1}{A}\right)^N \frac{JT}{4\pi^2}\int_{-\infty}^{\infty}\!\!\cdots\!\int_{-\infty}^{\infty} da_0\cdots da_N\,dp\,dq \exp\left\{-\frac{m}{2i\hbar\epsilon^2}\sum_{m=0}^{N}\right.$$

$$\times\left(4\sin^2\frac{m\pi}{2N} - \omega^2\epsilon^2\right)a_m^2 + ip\left[a_0+\sqrt{2}\sum_{m=1}^{N-1}a_m + a_N - \sqrt{T}x_a\right]$$

$$\left. + iq\left[a_0+\sqrt{2}\sum_{m=1}^{N-1}(-1)^m a_m + (-1)^N a_N - \sqrt{T}x_b\right]\right\} \tag{5.9}$$

where $\frac{1}{A} = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2}$ and $J = \frac{2}{(\epsilon)^{\frac{N+1}{2}}}$ is the jacobian of transformation (the derivation is presented in Appendix B).

Since the exponent can be separated into factors, the integrations can be performed separately. The results of such integrations are

$$\int_{-\infty}^{\infty} da_0 \exp\left\{ \frac{m\omega^2}{2i\hbar} a_0^2 + i(p+q)a_0 \right\} = \left[\frac{-2\pi i\hbar}{m\omega^2}\right]^{1/2} \exp\left\{ \frac{i\hbar}{2m\omega^2}(p+q)^2 \right\} \tag{5.10}$$

$$\int_{-\infty}^{\infty} da_N \exp\left\{ \frac{-m}{2i\hbar\epsilon^2}[4-\omega^2\epsilon^2]a_N^2 + i(p+(-1)^N q)a_N \right\}$$
$$= \left[\frac{2\pi i\hbar\epsilon^2}{m[4-\omega^2\epsilon^2]}\right]^{1/2} \exp\left\{ \frac{i\hbar\epsilon^2(p+(-1)^N q)^2}{2m[4-\omega^2\epsilon^2]} \right\} \tag{5.11}$$

$$\int_{-\infty}^{\infty} da_{n\text{-ev.}} \exp\left\{ \frac{-m}{2i\hbar\epsilon^2}\left[4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2\right]a_{n\text{-ev.}}^2 + \sqrt{2}\, i(p+q)a_{n\text{-ev.}} \right\}$$
$$= \left[\frac{2\pi i\hbar\epsilon^2}{m(4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2)}\right]^{1/2} \exp\left\{ \frac{-i\hbar(p+q)^2}{m[4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2]} \right\}_{n\text{-ev.}} \tag{5.12}$$

$$\int_{-\infty}^{\infty} da_{n\text{-odd}} \exp\left\{ \frac{-m}{2i\hbar\epsilon^2}\left[4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2\right]a_{n\text{-odd}}^2 + \sqrt{2}\, i(p-q)a_{n\text{-odd}} \right\}$$
$$= \left[\frac{2\pi i\hbar\epsilon^2}{m(4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2)}\right]^{1/2} \exp\left\{ \frac{-i\hbar(p-q)^2}{m[4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2]} \right\}_{n\text{-odd}} \tag{5.13}$$

Therefore, eq. (5.9) can be written as

$$K(x_b,T;x_a,0) = \lim_{N\to\infty} J\left(\frac{1}{A}\right)^N \frac{T}{4\pi^2}\left[\frac{-2\pi i\hbar}{m\omega^2}\right]^{1/2}\left[\frac{2\pi i\hbar\epsilon^2}{m[4-\omega^2\epsilon^2]}\right]^{1/2}\prod_{n=1}^{N-1}\left[\frac{2\pi i\hbar\epsilon^2}{m\left[4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2\right]}\right]^{1/2}$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp\,dq \exp\left\{ -i\sqrt{T}x_a p - i\sqrt{T}x_b q + \frac{i\hbar\epsilon^2(p+(-1)^N q)^2}{2m[4-\omega^2\epsilon^2]} \right.$$
$$- \frac{i\hbar\epsilon^2}{m}(p+q)^2 \sum_{n\text{-ev.}}\left[\frac{1}{4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2}\right]$$
$$\left. -\frac{i\hbar\epsilon^2}{m}(p-q)^2 \sum_{n\text{-odd}}\left[\frac{1}{4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2}\right] \right\} \tag{5.14}$$

Before performing the integration over p and q we take the limit of N approaching infinity in both sums in eq. (5.14) (this will not effect the results of the integration). As N approaches infinity we get

$$\sum_{n\text{-ev.}} \left[\frac{1}{4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2}\right] \longrightarrow \sum_{m=1}^{\infty} \left[\frac{1}{\frac{4m^2\pi^2}{N^2} - \omega^2\epsilon^2}\right] \qquad (5.15)$$

and

$$\sum_{n\text{-odd}} \left[\frac{1}{4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2}\right] \longrightarrow \sum_{m=1}^{\infty} \left[\frac{1}{\frac{(2m-1)^2\pi^2}{N^2} - \omega^2\epsilon^2}\right] \qquad (5.16)$$

Using the identities

$$\sum_{m=1}^{\infty} \left[\frac{1}{m^2 - x^2}\right] = \frac{1}{2x^2} - \frac{\pi}{2x}\cot \pi x \qquad (5.17)$$

and

$$\sum_{m=1}^{\infty} \left[\frac{1}{(2m-1)^2 - x^2}\right] = \frac{\pi}{4x}\tan\frac{\pi x}{2} \qquad (5.18)$$

we have

$$-\frac{i\hbar\epsilon^2(p+q)^2}{m}\sum_{n\text{-ev.}} \left[\frac{1}{4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2}\right] = -\frac{i\hbar T^2(p+q)^2}{4m}\left[\frac{2}{\omega^2 T^2} - \frac{1}{\omega T}\cot\frac{\omega T}{2}\right] \qquad (5.19)$$

and

$$-\frac{i\hbar\epsilon^2(p-q)^2}{m}\sum_{n\text{-odd}} \left[\frac{1}{4\sin^2\frac{n\pi}{2N} - \omega^2\epsilon^2}\right] = -\frac{i\hbar T^2(p-q)^2}{4m}\,\frac{1}{4\omega T}\tan\frac{\omega T}{2} \qquad (5.20)$$

as N approaches infinity.

The integral part of the eq. (5.14) can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp\,dq \exp\left\{ -i\sqrt{T}x_a p - i\sqrt{T}x_b q + \frac{i\hbar T}{4m\omega}\cot\frac{\omega T}{2}(p+q)^2 - \frac{i\hbar T}{4m\omega}\tan\frac{\omega T}{2}(p-q)^2 \right\}$$

$$= \left[\frac{2\pi m\omega}{\hbar T}\right]\exp\left\{ \frac{im\omega}{2\hbar\sin\omega T}\left[(x_a^2+x_b^2)\cos\omega T - 2x_a x_b \right]\right\} \qquad (5.21)$$

The eq. (5.14) then becomes

$$K(x_b,T;x_a,0) = \lim_{N\to\infty}\left(\frac{1}{A}\right)^N J\,\frac{T}{4\pi^2}\left[\frac{2\pi\hbar\epsilon}{m\omega}\right]\left[\frac{1}{4-\omega^2\epsilon^2}\right]^{1/2}\prod_{n=1}^{N-1}\left[\frac{2\pi i\hbar\epsilon^2}{m\left(4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right)}\right]^{1/2}$$

$$\times\left[\frac{2\pi m\omega}{\hbar T}\right]\exp\left\{\frac{im\omega}{2\hbar\sin\omega T}\left[(x_a^2+x_b^2)\cos\omega T - 2x_a x_b\right]\right\} \qquad (5.22)$$

On substituting $(1/A) = (m/2\pi i\hbar\epsilon)^{1/2}$ and $J = 2/(\epsilon)^{\frac{N+1}{2}}$ we obtain the prefactor $F(T)$;

$$F(T) = \lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\epsilon}\right]^{1/2}\frac{2}{[4-\omega^2\epsilon^2]^{1/2}}\prod_{n=1}^{N-1}\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]^{-1/2} \qquad (5.23)$$

The product in the eq. (5.23) can be written as

$$\prod_{n=1}^{N-1}\left[4\sin^2\frac{n\pi}{2N}-\omega^2\epsilon^2\right]^{-1/2} = \prod_{n=1}^{N-1}\left[4\sin^2\frac{n\pi}{2N}\right]^{-1/2}\prod_{n=1}^{N-1}\left[1-\frac{\omega^2\epsilon^2}{4\sin^2\frac{n\pi}{2N}}\right]^{-1/2} \qquad (5.24)$$

$$= \prod_{n=1}^{N-1}\left[2-2\cos\frac{n\pi}{N}\right]^{-1/2}\prod_{n=1}^{N-1}\left[1-\frac{\omega^2\epsilon^2}{4\sin^2\frac{n\pi}{2N}}\right]^{-1/2} \qquad (5.25)$$

Using the identity

$$\prod_{m=1}^{N-1}\left[x^2 - 2x\cos\frac{m\pi}{N} + 1\right] = \frac{x^{2N}-1}{x^2-1} \qquad (5.26)$$

we get

$$\prod_{m=1}^{N-1}\left[2 - 2\cos\frac{m\pi}{N}\right]^{-1/2} = \frac{1}{\sqrt{N}} \qquad (5.27)$$

The eq. (5.25) then becomes

$$\prod_{m=1}^{N-1}\left[4\sin^2\frac{m\pi}{2N} - \omega^2\epsilon^2\right]^{-1/2} = \frac{1}{\sqrt{N}}\prod_{m=1}^{N-1}\left[1 - \frac{\omega^2T^2}{m^2\pi^2}\right]^{-1/2} \qquad (5.28)$$

and eq. (5.23) becomes

$$F(T) = \lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar T}\right]^{1/2}\frac{2}{\left[4-\omega^2\epsilon^2\right]^{1/2}}\prod_{m=1}^{N-1}\left[1 - \frac{\omega^2\epsilon^2}{4\sin^2\frac{m\pi}{2N}}\right]^{-1/2} \qquad (5.29)$$

As N approaches infinity, $\sin\left(\frac{m\pi}{2N}\right)$ approaches $\frac{m\pi}{2N}$;
Therefore eq. (5.29) becomes

$$F(T) = \left[\frac{m}{2\pi i\hbar T}\right]^{1/2}\prod_{m=1}^{\infty}\left[1 - \frac{\omega^2T^2}{m^2\pi^2}\right]^{-1/2} \qquad (5.30)$$

Using the identity

$$\prod_{m=1}^{\infty}\left[1 - \frac{x^2}{m^2\pi^2}\right] = \frac{\sin x}{x} \qquad (5.31)$$

the prefactor in eq. (5.30) can be written as

$$F(T) = \left[\frac{m}{2\pi i\hbar T}\right]^{1/2}\left[\frac{\omega T}{\sin m\omega T}\right]^{1/2} = \left[\frac{m\omega}{2\pi i\hbar \sin m\omega T}\right]^{1/2} \qquad (5.32)$$

On substituting eq. (5.32) into eq. (5.22) we obtain

$$K(x_b,T;x_a,0) = \left[\frac{m\omega}{2\pi i\hbar \sin m\omega T}\right]^{1/2} \exp\left\{\frac{im\omega}{2\hbar \sin m\omega T}\left[(x_a^2+x_b^2)\cos\omega T - 2x_a x_b\right]\right\} \qquad (5.33)$$

We can see that by using the present technique the prefactor and the exponent of the classical action can be obtained simultaneously.

5.3 Conclusions and Discussions

In the present technique we have put the boundary points into the path integral and represent the paths as a cosine series. We restrict ourselves to the discrete-time assumption and transform the path integral to be the multiple integrals of the coefficients of the series. After performing the integrations and taking care of all factors we obtain the product series and the sum of the exponents. Finally we take the limit N approaches infinity and find that the product series converges to be the prefactor and the sumes of the exponent converge to be the classical action.

Since Feynman and Devies did not use discrete-time assumption they had to calculate the action function by direct integration. Feynman used the free-particle limit to calculate the prefactor, while Devies could not obtain the prefactor. Mathematically, this is due to the fact that the limit N approaches infinity was not properly taken in his paper. In our calculation we use the summation for calculating the action function and the limit N approaches infinity is taken at the final step. The prefactor and the classical action could be therefore obtained directly.