CHAPTER 4

Devies ' Method

In this chapter we present a method for calculating the harmonic-oscillator propagator as proposed by Devies. In the first section we discuss briefly the basic ideas of this method. In sec. 4.2 we show how the propagator can be obtained and we discuss the mathematical limitations of this method in the last section.

4.1 The Basic Ideas of Devies'Method

In the previous chapter we discussed the Feynman's method in which the propagator was obtained by calculating the prefactor and the classical action, separately. Instead of treating the problem in that way, Devies tried to obtain the prefactor and the exponent of the classical action simultaneously. In order to do this, he rewrote the path integral in the following form :

$$K(x_b,T;x_a,0) = \int \mathcal{D}[x(t)] \exp\left\{ \frac{i}{\hbar} \int_0^T L(\dot{x},x,t)\,dt \right\} \delta(x(0)-x_a)\,\delta(x(T)-x_b) \tag{4.1}$$

where L is the Lagrangian of the system given by the eq. (3.1) and Dirac delta functions indicate the constrain of the boundary points.

Following Feynman's idea of representing the paths as a Fourier series, he chose to represent the paths as a cosine series ;

$$x(t) = \sum_{n=0}^{N} b_n \cos \frac{n\pi t}{T} \tag{4.2}$$

The action function can then be written as

$$S = -\frac{T}{2} m\omega^2 b_0^2 + \frac{T}{4} m \sum_{n=1}^{N} \left[\left(\frac{n\pi}{T}\right)^2 - \omega^2 \right] b_n^2 \tag{4.3}$$

The path integral in eq. (4.1) can be written as

$$\begin{aligned} K(x_b,T;x_a,0) = {} & \lim_{N\to\infty} J\left(\frac{1}{A}\right)^N \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} db_0 \ldots db_n \exp\left\{ \frac{i}{\hbar}\left[-\frac{T}{2} m\omega^2 b_0^2 \right.\right. \\ & \left.\left. + \frac{T}{4} m \sum_{n=1}^{N} \left[\left(\frac{n\pi}{T}\right)^2 - \omega^2 \right] b_n^2 \right\} \delta(b_0 + b_1 + \cdots \\ & + b_N - x_a)\, \delta(b_0 - b_1 + \cdots + (-1)^N b_N - x_b) \end{aligned} \tag{4.4}$$

where J is the jacobian of transformation.

4.2 Performing the Integration

In order to perform integrating the eq. (4.4), Devies expressed eq. (4.4) in the following form :

$$\begin{aligned} K(x_b,T;x_a,0) = {} & \lim_{N\to\infty} J\left(\frac{1}{A}\right)^N \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} db_0 \ldots db_N\, dp\, dq \exp\left\{ -ipx_a \right. \\ & -iqx_b + \frac{i}{\hbar}\left[-\frac{T}{2} m\omega^2 b_0^2 + \frac{T}{4} m \sum_{n=1}^{N} \left[\left(\frac{n\pi}{T}\right)^2 - \omega^2 \right] b_n^2 \right] \\ & \left. + i\left[b_0(p+q) + b_1(p-q) + \cdots \right] \right\} \end{aligned} \tag{4.5}$$

He then performed integrations over b_0, $b_1, \ldots, b_N$ and obtained

$$K(x_b,T;x_a,0) = B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp\,dq\,\exp\Big\{-ipx_a - iqx_b + \frac{i\hbar}{2m\omega^2T}(p+q)^2 - \frac{i\hbar}{mT}(p+q)^2\sum_{n=1}^{N}\Big[\Big(\frac{2n\pi}{T}\Big)^2-\omega^2\Big]^{-1} + \frac{i\hbar}{mT}(p-q)^2\sum_{n=1}^{N}\Big[\frac{(2n-1)^2\pi^2}{T^2}-\omega^2\Big]^{-1}\Big\} \tag{4.6}$$

where B is some constant.

Using the identities

$$\frac{1}{m\omega^2T} + \frac{2}{mT}\sum_{n=1}^{\infty}\Big[\omega^2-\Big(\frac{2n\pi}{T}\Big)^2\Big]^{-1} = \frac{1}{2m\omega}\frac{\sin\omega T}{(1-\cos\omega T)} \tag{4.7}$$

and

$$\frac{1}{mT}\sum_{n=1}^{\infty}\Big[\omega^2-\frac{(2n-1)^2\pi^2}{T^2}\Big]^{-1} = -\frac{1}{4m\omega}\frac{\sin\omega T}{(1+\cos\omega T)} \tag{4.8}$$

the eq. (4.6) can be written as

$$K(x_b,T;x_a,0) = B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp\,dq\,\exp\Big\{-ipx_a - iqx_b + \frac{i\hbar}{4m\omega}(p+q)^2\cot\frac{\omega T}{2} - \frac{i\hbar}{4m\omega}(p-q)^2\tan\frac{\omega T}{2}\Big\} \tag{4.9}$$

Performing the integrations over p and q he obtained

$$K(x_b,T;x_a,0) = C\exp\left\{-\frac{im\omega}{\hbar}\left[(x_a+x_b)^2\tan\frac{\omega T}{2} - (x_a-x_b)^2\cot\frac{\omega T}{2}\right]\right\}$$

$$= C\exp\left\{\frac{im\omega}{2\hbar\sin\omega T}\left[(x_a^2+x_b^2)\cos\omega T - 2x_bx_a\right]\right\} \qquad (4.10)$$

where C is a normalizing constant, the modulus of which is determined by the requirement that $K(x_b, T ; x_a, 0)$ be the kernel of a unitary transformation. He did not calculate the constant C. However, by using this method, the exponent of the classical action can be correctly obtained

4.3 Conclusion and Discussion

Devies calculated the harmonic-oscillator propagator by putting the boundary points into the path integral and expressing the paths as a cosine series. Following Feynman , Devies transformed the path integral to be the multiple integrals of the coefficients of the series. He was able to obtain the exponent of the classical action after performing the integration. He remarked on the prefactor but did not perform any calculation.

In the following chapters we shall show that by combining Devies' and Feynmen's ideas the prefactor and the exponent of the classical action can be obtained simultaneously.