CHAPTER 2

Qualitative Survey

In this chapter we present the basic idea of the path integrals. As an introduction to the mathematical formalism of the path integral, we construct the path integral in the manner used by Feynman and Hibbs in their books.

2.1 Feynman's Path Integral

Let us consider the motion of a particle starting from one point to another. In quantum mechanics, we cannot specify its exact position at a given time we can only indicate the probability of finding it at any given time and place. Therefore there will be many possible paths that the particle can take. However, classically, we can specify that particle is moving along a particular path for which the action $S = \int_0^T L\,dt$ is minimum (where L is the lagranginan of the system). For simplicity, we shall restrict ourselves to the case of the particle moving in one dimension with the position at any given time specified by a coordinate x and the path x(t). If the particle at an initial time t_a starts from the point x_a and goes to a final point x_b at time t_b, then the classical path $\bar{x}(t)$ is that for which the principle of least action is satisfied, i.e. $S = 0$, and the classical action is given by

$$S_{cl} = \int_{t_a}^{t_b} L(\dot{\bar{x}}(t), \bar{x}(t), t)\,dt \qquad (2.1)$$

As we mentioned, in the quantum mechanical picture, there are many possible paths which the particle can take due to uncertainity inherent in the probabilistic interpretation of Quantum Mechanics. The probability that the particle goes from a point x_a at time t_a to the point x_b at time t_b is given by the absolute square of an amplitude $K(x_b, t_b; x_a, t_a)$ which is the sum of the contributions $\phi\,[x(t)]$, one from each path $x(t)$, viz.,

$$K(x_b, t_b; x_a, t_a) = \sum_{\substack{\text{over all paths} \\ \text{from } x_a \text{ to } x_b}} \phi[x(t)] \tag{2.2}$$

where the contribution of a path $x(t)$ has a phase proportional to the action s,

$$\phi[x(t)] = \text{const.}\ e^{\frac{i}{\hbar} S[x]} \tag{2.3}$$

To determine the probability amplitude $K(x_b, t_b ; x_a, t_a)$, we have to compute the sum in eq. (2.2) over all paths of infinite number, and it becomes appropriate to replace the infinite summation by path integration. In order to do this, we first choose a subset of all paths by dividing the independent time variable into a series of small intervals ϵ . This gives a set of values $t_a, t_1, t_2, \ldots,$ t_{N-1}, t_b where $\epsilon = t_{i+1} - t_i$, $N\epsilon = t_b - t_a$, and $t_N = t_b$. At each time t_i we select some special point x_i with $x_o = x_a$, $x_N = x_b$ and then construct a path by connecting all the points with straight line as shown in Fig. 1.

A sum over all paths constructed in this manner can be defined by taking a multiple integral over all values of x_i for i between 1 and N-1 so that the propagator in eq. (2.2) becomes

$$K(x_b,t_b;x_a,t_a) \sim \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi[x(t)]\,dx_1\ldots dx_{N-1} \qquad (2.4)$$

where the integrations over x_o and x_N are not involved since these are the fixed end points x_a and x_b. A more representative sample of the complete set of all possible paths between x_a and x_b can be obtained by making ϵ infinitesimal and by introducing some normalizing factor, which is expected to depend on ϵ, into eq. (2.4). The probability amplitude would then be written as

$$K(x_b,t_b;x_a,0) = \lim_{\epsilon\to 0}\frac{1}{A}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{\frac{i}{\hbar}S[x]\right\}\frac{dx_1}{A}\cdots\frac{dx_{N-1}}{A} \qquad (2.5)$$

where $\frac{1}{A} = \left[\frac{m}{2\pi i\hbar\epsilon}\right]^{1/2}$

The integration in eq. (2.5) can be written in a less restrictive notation as

$$K(x_b,t_b;x_a,t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}[x(t)]\exp\left\{\frac{i}{\hbar}S[x]\right\} \qquad (2.6)$$

which is called a "path integral", and the probability amplitude of this form is known as the Feynman propagator.

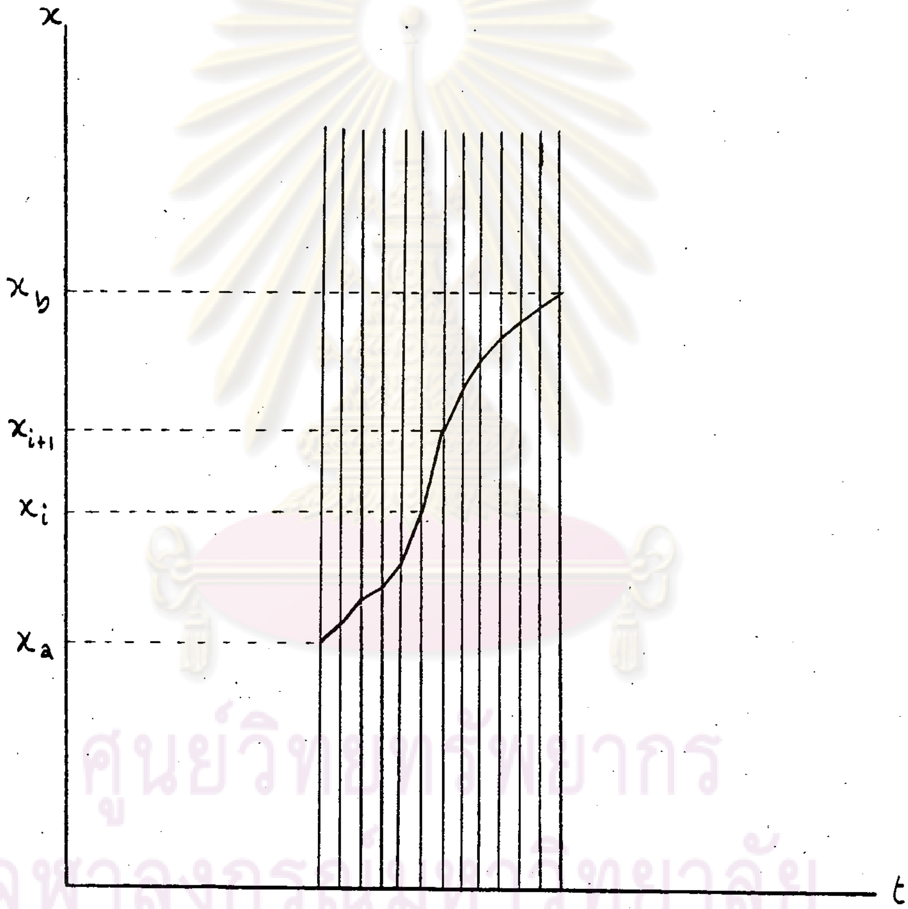

Fig. 1 The construction of the path integral