กึ่งมอดูลย่อย ๒-แอบซอร์บิงและกึ่งมอดูลย่อย ๒-แอบซอร์บิงอย่างอ่อนบนกึ่งริงสลับที่

นางสาวอิสราภรณ์ ทองสมนึก

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2555

2-ABSORBING AND WEAKLY 2-ABSORBING SUBSEMIMODULES OVER COMMUTATIVE SEMIRINGS

Miss Issaraporn Thongsomnuk

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2012

Thesis Title	2-ABSORBING AND WEAKLY 2-ABSORBING SUBSEMIMODULES OVER COMMUTATIVE SEMIRINGS
By	Miss Issaraporn Thongsomnuk
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Sajee Pianskool, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

.... Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

.... Chairman

(Associate Professor Amorn Wasanawichit, Ph.D.)

.... Thesis Advisor

(Assistant Professor Sajee Pianskool, Ph.D.)

.... Examiner

(Ouamporn Phuksuwan, Ph.D.)

.... External Examiner

(Ruangvarin Intarawong Sararnrakskul, Ph.D.)

อิสราภรณ์ ทองสมนึก : กึ่งมอดูลย่อย ๒-แอบซอร์บิงและกึ่งมอดูลย่อย ๒-แอบซอร์บิงอย่างอ่อนบนกึ่งริงสลับที่ (2-ABSORBING AND WEAKLY 2-ABSORBING SUBSEMIMODULES OVER COMMUTATIVE SEMIRINGS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก: ผศ.ดร.ศจี เพียรสกุล, 49หน้า.

ในวิทยานิพนธ์ฉบับนี้ ศึกษากึ่งมอดูลย่อย ๒-แอบซอร์บิง, กึ่งมอดูลย่อย ๒-แอบซอร์บิงอย่างอ่อน, กึ่งมอดูลย่อยเกือบวางนัยทั่วไป ๒-แอบซอร์บิงและกึ่งมอดูลย่อยเกือบวางนัยทั่วไป ๒-แอบซอร์บิงอย่างอ่อนของกึ่งมอดูลบนกึ่งริงสลับที่ แนวคิดเหล่านี้ถูกขยายมาจากกึ่งมอดูลย่อยเฉพาะ, กึ่งมอดูลย่อยเฉพาะอย่างอ่อน, กึ่งมอดูลย่อยปฐมภูมิและกึ่งมอดูลย่อยปฐมภูมิอย่างอ่อนตามลำดับ งานวิจัยนี้ให้การจำแนกลักษณะเฉพาะบางประการของสิ่งที่ได้กล่าวไปข้างต้น นอกจากนี้ ศึกษาความสัมพันธ์ระหว่างกึ่งมอดูลย่อย ๒-แอบซอร์บิง, กึ่งมอดูลย่อย ๒-แอบซอร์บิงอย่างอ่อน, กึ่งมอดูลย่อยเกือบวางนัยทั่วไป ๒-แอบซอร์บิงและกึ่งมอดูลย่อยเกือบวางนัยทั่วไป ๒-แอบซอร์บิงอย่างอ่อนของกึ่งมอดูลบนกึ่งริงสลับที่ กับไอดีล ๒-แอบซอร์บิง, ไอดีล ๒-แอบซอร์บิงอย่างอ่อน, ไอดีลเกือบวางนัยทั่วไป ๒-แอบซอร์บิงและไอดีลเกือบวางนัยทั่วไป ๒-แอบซอร์บิงอย่างอ่อนของกึ่งริงเดียวกัน ท้ายสุดนี้ได้เงื่อนไขที่จำเป็นและเพียงพอของกึ่งมอดูลที่จะเป็นกึ่งมอดูลบน

ภาควิชา คณิตศาสตร์และ วิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต..

ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก.....................

สาขาวิชา คณิตศาสตร์

ปีการศึกษา 2555

5472166123 : MAJOR MATHEMATICS
KEYWORDS : 2-ABSORBING SUBSEMIMODULE/ WEAKLY 2-ABSORBING SUBSEMIMODULE/ AG2-ABSORBING SUBSEMIMODULE/ WEAKLY AG2-ABSORBING SUBSEMIMODULE/ ZARISKI TOPOLOGY

ISSARAPORN THONGSOMNUK : 2-ABSORBING AND WEAKLY 2-ABSORBING SUBSEMIMODULES OVER COMMUTATIVE SEMIRINGS. ADVISOR : ASST. PROF. SAJEE PIANSKOOL, Ph.D., 49pp.

In this thesis, we study 2-absorbing, weakly 2-absorbing, AG2-absorbing and weakly AG2-absorbing subsemimodules of semimodules over commutative semirings. Those are extended from prime, weakly prime, primary and weakly primary subsemimodules, respectively. Some characterizations of 2-absorbing, weakly 2-absorbing, AG2-absorbing and weakly AG2-absorbing subsemimodules are obtained. Moreover, we investigate relationships between 2-absorbing, weakly 2-absorbing, AG2-absorbing and weakly AG2-absorbing subsemimodules of semimodules over commutative semirings and 2-absorbing, weakly 2-absorbing, AG2-absorbing and weakly AG2-absorbing ideals of the same semirings. Finally, we obtain necessary and sufficient conditions of a semimodule in order to be a top semimodule.

Department : ...Mathematics and.....	Student's Signature : ....
...Computer Science...	Advisor's Signature : ....
Field of Study : ......Mathematics......	
Academic Year : ....2012.............	

ACKNOWLEDGEMENTS

In the completion of my Master Thesis, I am deeply indebted to my thesis advisor, Assistant Professor Dr. Sajee Pianskool, not only for coaching my research but also for broadening my academic vision. I would like to express my special thanks to my thesis committee: Associate Professor Dr. Amorn Wasanawichit, Dr. Ouamporn Phuksuwan and Dr. Ruangvarin Intarawong Sararnrakskul. Their suggestions and comments are my sincere appreciation. Moreover, I feel very thankful to all of my teachers who have taught me for my knowledge and skills. Also, I wish to express my thankfulness to my family and my friends for their encouragement throughout my study.

Finally, I would like to thank the Development and Promotion of Science and Technology Talents Project (DPST) for financial support throughout my undergraduate and graduate study.

CONTENTS

page

ABSTRACT IN THAI iv

ABSTRACT IN ENGLISH v

ACKNOWLEDGEMENTS vi

CONTENTS vii

CHAPTER

I INTRODUCTION 1

II PRELIMINARIES 5

III 2-ABSORBING SUBSEMIMODULES AND WEAKLY 2-ABSORBING SUBSEMIMODULES OVER COMMUTATIVE SEMIRINGS 18

IV AG2-ABSORBING SUBSEMIMODULES AND WEAKLY AG2-ABSORBING SUBSEMIMODULES OVER COMMUTATIVE SEMIRINGS 38

REFERENCES 48

VITA 49

CHAPTER I
INTRODUCTION

In 2003, Anderson and Smith introduced the concept of weakly prime ideals of a commutative ring in [1]. They defined a ***weakly prime ideal*** P of a commutative ring R to be a proper ideal and if whenever $a, b \in R$ with $0 \neq ab \in P$, then $a \in P$ or $b \in P$.

In 2007, Atani and Farzalipour introduced the concept of weakly prime submodules over a commutative ring in [4]. They defined a ***weakly prime submodule*** N of an R-module M to be a proper submodule and if whenever $a \in R$ and $m \in M$ with $0 \neq am \in N$, then $m \in N$ or $a \in (N : M)$. In the same year, Badawi generalized the concept of prime ideals of a commutative ring to 2-absorbing ideals of a commutative ring in [6]. He defined a 2***-absorbing ideal*** I of a commutative ring R to be a proper ideal and if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Moreover, Badawi and Darani generalized the concept of weakly prime ideals to weakly 2-absorbing ideals in [7]. They defined a ***weakly*** 2***-absorbing ideal*** I of a commutative ring R to be a proper ideal and if whenever $a, b, c \in R$ with $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

In 2011, 2-absorbing and weakly 2-absorbing submodules have been introduced and studied by Darani and Soheilnia, see [12]. A proper submodule N of an R-module M is said to be a 2***-absorbing submodule* of** M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$ and a proper submodule N of an R-module M is said to be a ***weakly*** 2***-absorbing submodule*** *of* M if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$.

In 2012, Chaudhari introduced the concept of 2-absorbing ideals of a commutative semiring in [8]. He defined a 2***-absorbing ideal*** I of a commutative semiring R to be a proper ideal and if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

Since a ring R is also an R-module, the concept of (weakly) prime and (weakly) 2-absorbing ideals of rings are the special case of the concept of (weakly) prime and (weakly) 2-absorbing submodules. Similarly, a semiring R is an R-semimodule. Moreover, semimodules are generalization of modules. This leads us to extend the idea of 2-absorbing ideals of commutative semirings and (weakly) 2-absorbing submodules over a commutative ring to (weakly) 2-absorbing subsemimodules over a commutative semiring. In this research, we define a ***2-absorbing subsemimodule*** N of a semimodule M over a commutative semiring R to be a proper subsemimodule and if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$. And we define a proper subsemimodule N of a semimodule M over a commutative semiring R to be a ***weakly 2-absorbing subsemimodule*** of M if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$. Moreover, we define a ***weakly 2-absorbing ideal*** I of a commutative semiring R to be a proper ideal and if whenever $a, b, c \in R$ with $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Then, we study some characterizations of 2-absorbing and weakly 2-absorbing subsemimodules of semimodules over commutative semirings. In addition, we provide some ralationships between being (weakly) 2-absorbing subsemimodules of semimodules over commutative semirings and being (weakly) 2-absorbing ideals of commutative semirings.

The inspiration of the next target of this research arose from the followings. In 2010, prime subsemimodules were studied in multiplication R-semimodules by Yesilot, Oral and Tekir in [14] and Atani and Kohan in [5]. An R-semimodule M is called a ***multiplication R-semimodule*** if for all subsemimodule N of M there exists an ideal I of R such that $N = IM$. Moreover, the product of subsemimodules are introduced. Let N and K be subsemimodules of a multiplication R-semimodule M with $N = IM$ and $K = JM$ for some ideals I and J of R. The ***product of*** N ***and*** K, denoted by NK, is defined by $NK = (IJ)M$. This definition is, of course, independent of the choices of ideals I and J. A multiplication R-semimodule is interesting because its elements are allowed to be multiplied in some sense. For this reason, we study 2-absorbing subsemimodules of multiplication R-semimodules by extending some results of prime subsemimodules of multiplication R-semimodules.

The collection of all prime subtractive subsemimodules of an R-semimodule is a topology studied by Atani, S.E., Atani, R.E. and Tekir, U. in 2011, see [3]. This topology is called the ***Zariski topology***. In this work, we are also interested in studying that the collection of all 2-absorbing subtractive subsemimodules of an R-semimodule does satisfy the Zariski topology. We call an R-semimodule satisfying the Zariski topology a ***top semimodule***.

The notion of primary ideals of a commutative semiring and primary subsemimodules of semimodules over a commutative semiring have been introduced and studied by Atani and Kohan in 2010, see [5]. They defined a ***primary ideal*** I of a commutative semiring R to be a proper ideal and if whenever $a, b \in R$ with $ab \in I$, then $a \in I$ or $b^k \in I$ for some $k \in \mathbb{N}$ and a ***primary subsemimodule*** N of an R-semimodule M to be a proper subsemimodule and if whenever $a \in R$ and $m \in M$ with $am \in N$, then $m \in N$ or $a^k \in (N : M)$ for some $k \in \mathbb{N}$. Later in 2011, Chaudhari and Bonde extended these to weakly primary ideals and weakly primary subsemimodules, respectively, see [10]. They defined a ***weakly primary ideal*** I of a commutative semiring R to be a proper ideal and if whenever $a, b \in R$ with $0 \neq ab \in I$, then $a \in I$ or $b^k \in I$ for some $k \in \mathbb{N}$ and a ***weakly primary subsemimodule*** N of an R-semimodule M to be a proper subsemimodule and if whenever $a \in R$ and $m \in M$ with $0 \neq am \in N$, then $m \in N$ or $a^k \in (N : M)$ for some $k \in \mathbb{N}$. Besides, in the same year, the idea of 2-absorbing and weakly 2-absorbing submodules of modules over a commutative ring have been introduced by Darani and Soheilnia in [12].

In this reseach, we also aim to study the notion that generalizes primary and weakly primary subsemimodules and ideals in the same way as prime and weakly prime subsemimodules and ideals are extended. We define an ***almost general-***

ized* 2*-absorbing subsemimodule N of an R-semimodule M to be a proper subsemimodule and if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$, a ***weakly almost generalized*** 2***-absorbing subsemimodule*** N of an R-semimodule M to be a proper subsemimodule and if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$, an ***almost generalized*** 2***-absorbing ideal*** I of a commutative semiring R to be a proper ideal and if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $(bc)^k \in I$ for some $k \in \mathbb{N}$ and a ***weakly almost generalized*** 2***-absorbing ideal*** I of a commutative semiring R to be a proper ideal and if whenever $a, b, c \in R$ with $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $(bc)^k \in I$ for some $k \in \mathbb{N}$. Some of our results are analogous to the results given in [5], [10] and [12].

For this thesis, we give in Chapter II some basic definitions, examples and some known results. In Chapter III, we study 2-absorbing and weakly 2-absorbing subsemimodules of semimodules over a commutative semiring and those of multiplication R-semimodules over a commutative semiring and 2-absorbing and weakly 2-absorbing ideals of commutative semirings. Moreover, we find necessary and sufficient conditions of an R-semimodule in order to make it be a top semimodule. Finally, in Chapter IV, we investigate almost generalized 2-absorbing and weakly almost generalized 2-absorbing subsemimodules and ideals.

CHAPTER II

PRELIMINARIES

In this chapter, we collect definitions, some notation, terminology and some known results which will be used for this thesis.

Let $\mathbb{Z}$ denote the set of all intergers, $\mathbb{Z}^+$ the set of all positive integers, $\mathbb{Z}^-$ the set of all negative integers, $\mathbb{N}$ the set of natural numbers (positive intergers), $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$, $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$ and $\mathbb{Z}_n = \left\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\right\}$ where $n \in \mathbb{N}$.

Definition 2.1. [13] A ***semiring*** is a nonempty set R on which the operations of addition and multiplication are defined such that the following conditions are satisfied:

(1) $(R, +)$ is a commutative monoid with identity element 0_R,

(2) $(R, \cdot)$ is a monoid with identity element 1_R (we write ab instead of $a \cdot b$ for all $a, b \in R$),

(3) the multiplication distributes over the addition from both sides, and

(4) $0_R r = 0 = r 0_R$ for all $r \in R$.

Definition 2.2. [13] An ***ideal*** of a semiring R is a nonempty subset I of R satisfying the following conditions:

(1) if $a, b \in I$, then $a + b \in I$, and

(2) if $a \in I$ and $r \in R$, then $ra \in I$ and $ar \in I$.

Definition 2.3. [13] Let R be a semiring. A ***left R-semimodule*** (or a ***left semimodule over*** R) is a commutative monoid $(M, +)$ with additive identity 0_M for which a function $R \times M \to M$, denoted by $(r, m) \mapsto rm$ and called the ***scalar multiplication***, satisfies the following conditions for all elements r and r' of R and all elements m and m' of M:

(1) $(rr')m = r(r'm)$,

(2) $r(m + m') = rm + rm'$,

(3) $(r + r')m = rm + r'm$,

(4) $1_R m = m$, and

(5) $r0_M = 0_M = 0_R m$.

A right R-semimodule is defined analogously to a left R-semimodule. We simply, sometimes, write 0 instead of 0_R and 0_M. In this thesis, all semirings are commutative with nonzero identity. A semiring $(R, +, \cdot)$ is ***commutative*** if $\cdot$ is commutative, i.e., $ab = ba$ for all $a, b \in R$. Moreover, by an R-semimodule we mean a left R-semimodule, i.e., a left semimodule over a commutative semiring R.

Example. (1) Semirings R are R-semimodules.

(2) Modules over a ring R are R-semimodules.

(3) Vector spaces over a field F are F-semimodules.

(4) Let $R = \mathbb{Z}_0^+$ and $M = 2\mathbb{Z}_0^+$. Then $(2\mathbb{Z}_0^+, +, \cdot)$ is an R-semimodule, which is not an R-module.

From the definitions of semirings and semimodules and above example, we see that every ring with identity is a semiring and every unital module is a semimodule. In other words, semirings and semimodules are generalization of rings with identity and unital modules, respectively.

Definition 2.4. [13] Let M be an R-semimodule and N a subset of M. We say that N is a ***subsemimodule*** of M percisely when N is itself an R-semimodule with respect to the operations for M.

Proposition 2.5. [13] *Let M be an R-semimodule and $\{N_i \mid i \in \Lambda\}$ a family of subsemimodules of M. Then $\bigcap N_i$ is a subsemimodule of M.*

Definition 2.6. [13] Let M be an R-semimodule. The set $\sum_{i\in\Lambda} N_i$ consists of all finite sums of elements of $\bigcup_{i\in\Lambda} N_i$ where N_i is a subsemimodule of M for all i.

Proposition 2.7. [13] *Let M be an R-semimodule. If N_i is a subsemimodule of M for all i, then $\sum_{i\in\Lambda} N_i$ is a subsemimodule of M which is the smallest subsemimodule of M containing each of the N_i.*

Proposition 2.8. *Let M be an R-semimodule, I and J ideals of R. Then $(IJ)M = I(JM)$.*

Proof. First, assume that $x \in (IJ)M$. Then there exist $a_i \in I$, $b_i \in J$ where $i \in \{1, 2, \ldots, n\}$ and $m \in M$ such that $x = (\sum_{i=1}^{n} a_i b_i)m$. Thus $x = (\sum_{i=1}^{n} a_i b_i)m = (a_1b_1)m + (a_2b_2)m + \cdots + (a_nb_n)m = a_1(b_1m) + a_2(b_2m) + \cdots + a_n(b_nm) \in I(JM)$.

Therefore $(IJ)M \subseteq I(JM)$.

Next, assume that $x \in I(JM)$. Thus $x = a(bm)$ for some $a \in I$, $b \in J$ and $m \in M$. Then $x = a(bm) = (ab)m \in (IJ)M$.

Therefore $I(JM) \subseteq (IJ)M$. □

Notation: [10] Let M be an R-semimodule, N a subsemimodule of M, A a nonempty subset of M and $m \in M$. Let

$$(N : A) = \{r \in R \mid rA \subseteq N\} \qquad \text{and}$$
$$(N : m) = \big(N : \{m\}\big) = \{r \in R \mid rm \in N\}.$$

Example. Let $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_6$, $N = \{\bar{0}, \bar{3}\}$ and $A = \{\bar{1}, \bar{2}\}$.
Then $(N : A) = \big(\{\bar{0}, \bar{3}\} : \{\bar{1}, \bar{2}\}\big) = 3\mathbb{Z}_0^+$.

Proposition 2.9. [13] *Let M be an R-semimodule, N and N' subsemimodules of M. If A is a nonempty subset of M, then $(N \cap N' : A) = (N : A) \cap (N' : A)$.*

Proposition 2.10. [13] *Let M be an R-semimodule, N a subsemimodule of M and $m \in M$. Then $(N : M)$ and $(N : m)$ are ideals of R.*

Proposition 2.11. *Let M be an R-semimodule and N a subsemimodule of M. Then N is a proper subsemimodule of M if and only if $(N : M)$ is a proper ideal of R.*

Proof. First, assume that N is a proper subsemimodule of M. Suppose that $(N : M) = R$. Thus $1 \in (N : M)$. Then $M \subseteq N$ contradicts the fact that N is a proper subsemimodule of M.

Therefore $(N : M)$ is a proper ideal of R.

Next, assume that $(N : M)$ is a proper ideal of R. Suppose that $N = M$. Since $RM \subseteq M$, we get that $R = (M : M) = (N : M)$ contradicts the fact that $(N : M)$ is a proper ideal of R.

Therefore N is a proper subsemimodule of M. □

Definition 2.12. [10] An ideal I of a semiring R is called a ***subtractive ideal*** (or k**-ideal**) if $a, a + b \in I$ and $b \in R$, then $b \in I$.

A subsemimodule N of an R-semimodule M is called a ***subtractive subsemimodule*** (or k**-subsemimodule**) if $x, x + y \in N$ and $y \in M$, then $y \in N$.

Example. (1) Let $R = \mathbb{Z}_0^+$. Consider $I = 2\mathbb{Z}_0^+$. Clearly, I is an ideal of R. Next, we show that I is subtractive. Let $a, a + b \in I$ and $b \in R$. Then $a = 2k$ and $a + b = 2k'$ for some $k, k' \in \mathbb{Z}_0^+$. Thus $2k + b = 2k'$. We get that $2(k' - k) = b$. Since $b \in \mathbb{Z}_0^+$, we have $k' - k \in \mathbb{Z}_0^+$. Hence $b \in 2\mathbb{Z}_0^+ = I$.

Therefore I is a subtractive ideal of R.

(2) Let M be an R-semimodule. Clearly, $\{0\}$ is a subsemimodule of M. Next, we show that $\{0\}$ is subtractive. Let $a, a + b \in \{0\}$ and $b \in M$. Then $a = 0$ and $a + b = 0$. Thus $b = 0 + b = a + b = 0$. Hence $b \in \{0\}$.

Therefore $\{0\}$ is always a subtractive subsemimodule of any R-semimodule M.

Proposition 2.13. [10] *Let M be an R-semimodule. If N is a subtractive subsemimodule of M and $m \in M$, then $(N : M)$ and $(N : m)$ are subtractive ideals of R.*

Definition 2.14. [13] Let M be an R-semimodule. The ***annihilator*** of M, denoted by $\text{ann}(M)$, is defined as $\text{ann}(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\} = \big(\{0\} : M\big)$.

Definition 2.15. [13] A ***faithful*** R-semimodule M is one where the scalar multiplication of each $r \neq 0$ in R on M is nontrivial (i.e. $rx \neq 0$ for some x in M).

Equivalently, an R-semimodule M is faithful if the annihilator of M is the zero ideal.

Definition 2.16. [12] Let R be a commutative semiring. A proper ideal P of R is said to be a ***prime ideal*** if whenever $a, b \in R$ with $ab \in P$, then $a \in P$ or $b \in P$.

A proper ideal P of R is said to be a ***weakly prime ideal*** if whenever $a, b \in R$ with $0 \neq ab \in P$, then $a \in P$ or $b \in P$.

A proper ideal P of R is said to be a ***primary ideal*** if whenever $a, b \in R$ with $ab \in P$, then $a \in P$ or $b^k \in P$ for some $k \in \mathbb{N}$.

A proper ideal P of R is said to be a ***weakly primary ideal*** if whenever $a, b \in R$ with $0 \neq ab \in P$, then $a \in P$ or $b^k \in P$ for some $k \in \mathbb{N}$.

Definition 2.17. [10] Let M be an R-semimodule. A proper subsemimodule N of M is said to be a ***prime subsemimodule*** if whenever $a \in R$ and $m \in M$ with $am \in N$, then $m \in N$ or $a \in (N : M)$.

A proper subsemimodule N of M is said to be a ***weakly prime subsemimodule*** if whenever $a \in R$ and $m \in M$ with $0 \neq am \in N$, then $m \in N$ or $a \in (N : M)$.

A proper subsemimodule N of M is said to be a ***primary subsemimodule*** if whenever $a \in R$ and $m \in M$ with $am \in N$, then $m \in N$ or $a^k \in (N : M)$ for some $k \in \mathbb{N}$.

A proper subsemimodule N of M is said to be a ***weakly primary subsemimodule*** if whenever $a \in R$ and $m \in M$ with $0 \neq am \in N$, then $m \in N$ or $a^k \in (N : M)$ for some $k \in \mathbb{N}$.

Definition 2.18. [8] Let R be a commutative semiring. A proper ideal I of R is said to be a ***2-absorbing ideal*** if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

Definition 2.19. [12] Let M be a module over a commutative ring R. A proper submodule N of M is said to be a ***2-absorbing submodule*** if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$.

A proper submodule N of M is said to be a ***weakly 2-absorbing submodule*** if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$.

One can see that (weakly) prime and (weakly) 2-absorbing ideals of a commutative ring are the special cases of (weakly) prime and (weakly) 2-absorbing submodules of a module over a commutative ring. Also known that any semiring R is an R-semimodule and semimodules are generalization of modules. These guided us to extend the idea of 2-absorbing ideals of a commutative semiring and (weakly) 2-absorbing submodules of a module over a commutative ring to weakly 2-absorbing ideals of a commutative semiring and (weakly) 2-absorbing subsemimodules of a semimodule over a commutative semiring, respectively. Moreover, we extend the concept of (weakly) 2-absorbing subsemimodules to (weakly) almost generalized 2-absorbing subsemimodules.

Definition 2.20. Let R be a commutative semiring. A proper ideal I of R is said to be a ***weakly* 2*-absorbing ideal*** if whenever $a, b, c \in R$ with $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

A proper ideal I of R is said to be an ***almost generalized* 2*-absorbing ideal*** (or ***AG*2*-absorbing ideal*** for short) if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $(bc)^k \in I$ for some $k \in \mathbb{N}$.

A proper ideal I of R is said to be a ***weakly almost generalized* 2*-absorbing ideal*** (or ***weakly AG*2*-absorbing ideal*** for short) if whenever $a, b, c \in R$ with $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $(bc)^k \in I$ for some $k \in \mathbb{N}$.

Further, while we have done the research, the concept of a weakly 2-absorbing ideal of a commutative semiring is defined in the same way that our definition by Darani in [11].

Definition 2.21. Let M be an R-semimodule. A proper subsemimodule N of M is said to be a ***2-absorbing subsemimodule*** if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$.

A proper subsemimodule N of M is said to be a ***weakly* 2*-absorbing subsemimodule*** if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$.

A proper subsemimodule N of M is said to be an ***almost generalized* 2*-absorbing subsemimodule*** (or ***AG*2*-absorbing subsemimodule*** for short) if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$.

A proper subsemimodule N of M is said to be a ***weakly almost generalized* 2*-absorbing subsemimodule*** (or ***weakly AG*2*-absorbing subsemimodule*** for short) if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$.

Remark. From the definitions, we obtain the followings.

(1) $\{0\}$ is weakly 2-absorbing and weakly AG2-absorbing subsemimodules.

(2) 2-absorbing subsemimodules are AG2-absorbing subsemimodules. But the converse does not necessary hold. For example, consider the case where $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_0^+$ and $N = 8\mathbb{Z}_0^+$. Let $x \in N$, i.e., $x = 8k$ for some $k \in \mathbb{Z}_0^+$. Consider $a, b \in R$ and $m \in M$ with $ab(m) = 8k \in N$. Then there are 5 ways to write $8k$ as a product of $ab(m)$ as follows: $2a_1 \cdot 2a_2 \cdot (2a_3)$; $4b_1 \cdot 1b_2 \cdot (2b_3)$; $8c_1 \cdot 1c_2 \cdot (1c_3)$; $2d_1 \cdot 1d_2 \cdot (4d_3)$ and $1e_1 \cdot 1e_2 \cdot (8e_3)$ where $0 \leq a_i, b_i, c_i, d_i, e_i \leq k$; $a_i, b_i, c_i, d_i, e_i \in \mathbb{Z}_0^+$ and $i \in \{1, 2, 3\}$.
If $2a_1 \cdot 2a_2 \cdot (2a_3) \in N$, then $(2a_1 \cdot 2a_2)^2 \in (N : M)$.
If $4b_1 \cdot 1b_2 \cdot (2b_3) \in N$, then $(4b_1 \cdot 1b_2)^2 \in (N : M)$.
If $8c_1 \cdot 1c_2 \cdot (1c_3) \in N$, then $8c_1 \cdot 1c_2 \in (N : M)$.
If $2d_1 \cdot 1d_2 \cdot (4d_3) \in N$, then $(2d_1 \cdot 1d_2)^3 \in (N : M)$.
If $1e_1 \cdot 1e_2 \cdot (8e_3) \in N$, then $1e_1 \cdot 8e_3 \in N$.
Then N is an AG2-absorbing subsemimodule of M which is not 2-absorbing because $2 \cdot 2 \cdot (2) \in N$ but $2 \cdot 2 \notin N$ and $2 \cdot 2 \notin (N : M)$.

(3) Weakly 2-absorbing subsemimodules are weakly AG2-absorbing subsemimodules. But the converse does not necessary hold. For example, consider the case where $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_{24}$ and $N = \{\bar{0}, \bar{8}, \overline{16}\}$. Then N is a weakly AG2-absorbing subsemimodule of M which is not weakly 2-absorbing because $2 \cdot 2 \cdot (\bar{2}) \in N$ but $2 \cdot \bar{2} \notin N$ and $2 \cdot 2 \notin (N : M)$.

(4) AG2-absorbing subsemimodules are weakly AG2-absorbing subsemimod-

ules. But the converse does not necessary hold. For example, consider the case where $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_{12}$ and $N = \{\bar{0}\}$. Then N is a weakly AG2-absorbing subsemimodule of M which is not AG2-absorbing because $2 \cdot 2 \cdot (\bar{3}) \in N$ but $2 \cdot \bar{3} \notin N$ and $(2 \cdot 2)^k \notin (N : M)$ for all $k \in \mathbb{N}$.

(5) 2-absorbing subsemimodules are weakly 2-absorbing subsemimodules. But the converse does not necessary hold. For example, consider the case where $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_{12}$ and $N = \{\bar{0}\}$. Then N is a weakly 2-absorbing subsemimodule of M. From (4), we know that N is not AG2-absorbing so that N is not 2-absorbing by (2).

From (1), we see that $\{0\}$ is always a weakly 2-absorbing subsemimodule, but it is not a 2-absorbing subsemimodule as shown in the following proposition.

Proposition 2.22. *If $R = \mathbb{Z}_0^+$ and $M = \mathbb{Z}_n$ where $n = pqr$; $p, q, r \in \mathbb{N}$ and $1 < p, q, r < n$, then the zero subsemimodule of M is not 2-absorbing.*

Proof. Assume that $R = \mathbb{Z}_0^+$ and $M = \mathbb{Z}_n$ where $n = pqr$ with $p, q, r \in \mathbb{N}$ and $1 < p, q, r < n$. Note that $pq(r) = n \in \{\bar{0}\}$ in $\mathbb{Z}_n$. Since $1 < pr, qr, pq < n$, it follows that $pr \notin \{\bar{0}\}$, $qr \notin \{\bar{0}\}$ and $pq \notin \big(\{\bar{0}\} : \mathbb{Z}_n\big)$.

Therefore $\{\bar{0}\}$ is not 2-absorbing. □

We see that weakly 2-absorbing subsemimodules are generalization of 2-absorbing subsemimodules.

Proposition 2.23. *Let M be an R-semimodule and N a subsemimodule of M.*

(i) If N is a prime subsemimodule, then N is a 2-absorbing subsemimodule.

(ii) If N is a weakly prime subsemimodule, then N is a weakly 2-absorbing subsemimodule.

Proof. (i) Assume that N is a prime subsemimodule. Let $a, b \in R$ and $m \in M$ with $abm \in N$, but $am \notin N$ and $bm \notin N$. We claim that $ab \in (N : M)$. Since $abm \in N$ and N is a prime subsemimodule, $m \in N$ or $ab \in (N : M)$. If $m \in N$, then $am \in N$ contradicts $am \notin N$. Thus $ab \in (N : M)$

Therefore N is a 2-absorbing subsemimodule.

(ii) The proof is similar to that of (i). □

From Proposition 2.23, we obtain that prime and weakly prime subsemimodules are 2-absorbing and weakly 2-absorbing subsemimodules, respectively. But the converse does not necessary hold. For example, consider the case where $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_0^+$ and $N = 6\mathbb{Z}_0^+$. Then N is a (weakly) 2-absorbing subsemimodule of M which is not weakly prime because $0 \neq 2 \cdot 3 \in N$ but $2 \notin N$ and $3 \notin (N : M)$. Thus N is not a prime subsemimodule. Therefore 2-absorbing and weakly 2-absorbing subsemimodules are indeed generalizations of prime and weakly prime subsemimodules.

Given a submodule of a module leads to a factor module. Then we are curious whether the construction of a factor semimodule can be made. Next, we study the construction of a factor semimodule.

Definition 2.24. [2] A subsemimodule N of an R-semimodule M is called a ***partitioning*** subsemimodule if there exists a nonempty subset Q of M such that

(1) $RQ \subseteq Q$ where $RQ = \{rq \mid r \in R \text{ and } q \in Q\}$,

(2) $M = \bigcup \{q + N \mid q \in Q\}$ where $q + N = \{q + n \mid n \in N\}$, and

(3) if $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \varnothing$ if and only if $q_1 = q_2$.

In general, a partitioning subsemimodule N via a nonempty subset Q is called a **Q-subsemimodule**.

The construction of a factor semimodule [2]

Let M be an R-semimodule and N a Q-subsemimodule of M. Let $M/N_{(Q)} = \{q + N \mid q \in Q\}$. Then $M/N_{(Q)}$ is a semimodule over R under the addition $\oplus$ and the scalar multiplication $\odot$ defined as follows: for any $q_1, q_2, q \in Q$ and $r \in R$,

$$(q_1 + N) \oplus (q_2 + N) = q_3 + N \qquad \text{and} \qquad r \odot (q + N) = q_4 + N$$

where $q_3, q_4 \in Q$ are the unique elements such that $q_1 + q_2 + N \subseteq q_3 + N$ and $rq + N \subseteq q_4 + N$. The R-semimodule $M/N_{(Q)}$ is called a ***factor semimodule***.

To see that $\oplus$ and $\odot$ are well-defined, by (2) of the definition of Q-subsemimodules, there are $q_3, q_4 \in Q$ such that $q_1 + q_2 + N \subseteq q_3 + N$ and $rq + N \subseteq q_4 + N$. For the uniqueness of q_3 and q_4, (3) of the same definition guarantees this. Suppose that there exist $q_3', q_4' \in Q$ such that $q_1 + q_2 + N \subseteq q_3' + N$ and $rq + N \subseteq q_4' + N$. Thus $(q_3 + N) \cap (q_3' + N) \neq \varnothing$ and $(q_4 + N) \cap (q_4' + N) \neq \varnothing$. Then we obtain that $q_3 = q_3'$ and $q_4 = q_4'$.

Since $M/N_{(Q)}$ is a semimodule, its zero element must exist. Let $q_0 + N$ be the zero element of $M/N_{(Q)}$. For every $q \in Q$, from (1) of Definition 2.24, we obtain that $0_M = 0_R q \in Q$. Consider $(0_M + N) \oplus (q_0 + N) = 0_M + N$ because $q_0 + N$ is the zero element. Thus 0_M must be the unique element in Q such that $0_M + q_0 + N \subseteq 0_M + N$. Then $q_0 + N \subseteq 0_M + N$. So $(q_0 + N) \cap (0_M + N) \neq \varnothing$. We can conclude that $0_M = q_0$. This shows that the zero element of the semimodule $M/N_{(Q)}$ is $0_M + N$.

Proposition 2.25. [10] *Let N be a Q-subsemimodule of an R-semimodule M. If $r \in R$ and $m \in M$, then there exists a unique $q \in Q$ such that $m \in q + N$ and $rm \in r \odot (q + N)$.*

Proposition 2.26. [9] *Let M be an R-semimodule, N a Q-subsemimodule of M and P a subtractive subsemimodule of M with $N \subseteq P$. Then the followings hold:*

(i) N is a $Q \cap P$-subsemimodule of P.

(ii) $P/N_{(Q \cap P)} = \{q + N \mid q \in Q \cap P\}$ is a subsemimodule of $M/N_{(Q)}$.

Remark. The zero element of $P/N_{(Q \cap P)}$ is the same as the zero element of $M/N_{(Q)}$ which is $0_M + N$.

Definition 2.27. [12] A subsemimodule N of an R-semimodule M is called a ***nilpotent subsemimodule*** if $(N : M)^k N = \{0\}$ for some $k \in \mathbb{N}$.

Definition 2.28. [13] An R-semimodule M is called ***cyclic*** if there exists an element $m \in M$ such that $M = Rm$.

Let consider ideals $(N : M)$ and $(N : m)$ where M is a cyclic R-semimodule. Let $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_6 = R\bar{1}$ and $N = \{\bar{0}\}$. Then $(N : M) = \big(\{\bar{0}\} : \mathbb{Z}_6\big) = 6\mathbb{Z}_0^+$, $(N : \bar{2}) = \big(\{\bar{0}\} : \bar{2}\big) = 3\mathbb{Z}_0^+$ and $(N : \bar{1}) = \big(\{\bar{0}\} : \bar{1}\big) = 6\mathbb{Z}_0^+$. This shows that $(N : R\bar{1}) \neq (N : \bar{2})$ but $(N : R\bar{1}) = (N : \bar{1})$. Notice that $\bar{1}$ is a generator of M.

Proposition 2.29. *Let M be a cyclic R-semimodule with $M = Rm$. Then the ideals $(N : M)$ and $(N : m)$ are identical.*

Proof. It is clear that $(N : M) \subseteq (N : m)$. Next, let $x \in (N : m)$. Then $xm \in N$. Thus $Rxm \subseteq RN \subseteq N$. So $xRm \subseteq N$ and then $xM \subseteq N$, i.e., $x \in (N : M)$. Hence $(N : m) \subseteq (N : M)$.

Therefore $(N : M) = (N : m)$ as desired. □

Definition 2.30. [14] Let M be an R-semimodule. We call M a ***multiplication R-semimodule*** if for all subsemimodule N of M there exists an ideal I of R such that $N = IM$ and I is called a ***presentation ideal*** of N.

Note: [14] If M is a multiplication R-semimodule and N a subsemimodule of M, then there exists an ideal I of R such that $N = IM$. Thus $I \subseteq (N : M)$. Then $N = IM \subseteq (N : M)M \subseteq N$ and therefore $N = (N : M)M$. This shows that one presentation ideal of a subsemimodule N of a multiplication R-semimodule M is $(N : M)$.

Proposition 2.31. *If M is a cyclic R-semimodule with generator m, then M is a multiplication R-semimodule.*

Proof. Assume that $M = Rm$ for some $m \in M$. Let N be a subsemimodule of M. We claim that $N = (N : m)Rm$. First, let $x \in N$. Then there exists $r \in R$ such that $x = rm$, i.e., $r \in (N : m)$. We obtain that $x = rm \in rRm \subseteq (N : m)Rm$. Hence $x \in (N : m)Rm$. Next, $n \in (N : m)Rm$. Then there exist $r \in (N : m)$ and $s \in R$ such that $n = rsm$. Since $r \in (N : m)$, we get that $rm \in N$. Then $n = rsm = srm \in sN \subseteq N$. Hence $n \in N$. Thus $N = (N : m)Rm = (N : m)M$ as claimed.

Therefore M is a multiplication R-semimodule. □

The following proposition shows that in order to verify an R-semimodule M is a multiplication R-semimodule it is sufficient to prove only that there exists a presentation ideal of each subsemimodule of the form Rm where $m \in M$.

Proposition 2.32. [14] *An R-semimodule M is a multiplication R-semimodule if and only if there exists an ideal I of R such that $Rm = IM$ for each $m \in M$.*

Definition 2.33. [14] Let M be a multiplication R-semimodule. Moreover, let N and K be subsemimodules of M with $N = IM$ and $K = JM$ for some ideals I and J of R. The ***product of*** N ***and*** K, denoted by NK, is defined by $NK = (IJ)M$.

For $m_1, m_2 \in M$, the product of Rm_1 and Rm_2 is $Rm_1Rm_2 = (I_1M)(I_2M) = (I_1I_2)M$ where I_1 and I_2 are presentation ideals of the subsemimodules Rm_1 and Rm_2, respectively. We write m_1m_2 instead of Rm_1Rm_2.

For a subsemimodule N of M, if $N = IM$ for some ideal I of R, then $N^n = I^nM$ for any $n \in \mathbb{N}$.

Theorem 2.34. [14] *The product of two subsemimodules is independent of their presentation ideals.*

Theorem 2.34 makes sure that the product of subsemimodules N and K of a multiplication R-semimodule is well-defined.

Definition 2.35. [13] Let J be an ideal of a commutative semiring R. Then the ***radical of*** J, denoted by $\sqrt{J}$, is defined to be the intersection of all prime ideals of R containing J.

For an ideal J of a commutative semiring R, one can show that the set $\{x \in R \mid x^n \in J \text{ for some } n \in \mathbb{N}\}$ is an ideal of R.

Proposition 2.36. [13] *If J is an ideal of a commutative semiring R, then $\sqrt{J}$ is, in fact, the ideal $\{x \in R \mid x^n \in J \text{ for some } n \in \mathbb{N}\}$.*

We can rewrite the condition for being (weakly) AG2-absorbing ideals and (weakly) AG2-absorbing subsemimodules as follows:

An AG2-absorbing ideal of a semiring R is a proper ideal I of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in \sqrt{I}$.

A weakly AG2-absorbing ideal of a semiring R is a proper ideal I of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in \sqrt{I}$.

An AG2-absorbing subsemimodule of an R-semimodule M is a proper subsemimodule N of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N : M)}$.

A weakly AG2-absorbing subsemimodule of an R-semimodule M is a proper subsemimodule N of M if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N : M)}$.

Definition 2.37. [10] Let N be a proper subsemimodule of a nonzero R-semimodule M. Then the ***M-radical of*** N, denoted by $\text{rad}_M(N)$, is defined to be the intersection of all prime subsemimodules of M containing N.

Since a semiring R is also an R-semimodule, the radical of an ideal N of R is the special case of the M-radical of N where N is a subsemimodule of an R-semimodule M, i.e., $\text{rad}_M(N) = \sqrt{N}$.

Proposition 2.38. [5] *If M is an R-semimodule and N is a subsemimodule of M, then $\sqrt{(N : M)}M \subseteq \text{rad}_M(N)$.*

CHAPTER III

2-ABSORBING SUBSEMIMODULES AND WEAKLY 2-ABSORBING SUBSEMIMODULES OVER COMMUTATIVE SEMIRINGS

In this chapter, we extend some definitions and results of [5], [6], [10], [12] and [14] of modules over a commutative ring to those of semimodules over a commutative semiring.

It is known that any intersections of prime subsemimodules are not necessary prime subsemimodules. The following theorem shows the result of any intersections of each pair of prime and weakly prime subsemimodules.

For the rest of this thesis, unless otherwise stated, let R be a commutative semiring with nonzero identity.

Theorem 3.1. *Let M be an R-semimodule.*

(i) The intersection of each pair of distinct prime subsemimodules of M is a 2-absorbing subsemimodule of M.

(ii) The intersection of each pair of distinct weakly prime subsemimodules of M is a weakly 2-absorbing subsemimodule of M.

Proof. (i) Let N and K be two distinct prime subsemimodules of M. Then $N \cap K$ is a proper subsemimodule of M. Assume that $a, b \in R$ and $m \in M$ with $abm \in N \cap K$, but $am \notin N \cap K$ and $bm \notin N \cap K$. Then $abm \in N$ and $abm \in K$. We claim that $ab \in (N \cap K : M)$.

Case 1: $am \notin N$ and $bm \notin N$. Since $abm \in N$ and N is a prime subsemimodule, $bm \in N$ or $a \in (N : M)$. So $a \in (N : M)$ because $bm \notin N$. Thus $aM \subseteq N$ so that $am \in N$ contradicts $am \notin N$. Hence Case 1 is impossible.

Case 2: $am \notin K$ and $bm \notin K$. The proof is similar to that of Case 1 so that Case 2 is impossible.

Case 3: $am \notin N$ and $bm \notin K$. Since $b(am) = abm \in N$ which is a prime subsemimodule, $am \in N$ or $b \in (N : M)$. Thus $b \in (N : M)$ because $am \notin N$. We obtain that $bM \subseteq N$. Then $abM = baM \subseteq bM \subseteq N$. Thus $ab \in (N : M)$. The fact that $abm \in K$ also leads to $ab \in (K : M)$. Thus $ab \in (N : M) \cap (K : M) = (N \cap K : M)$.

Case 4: $am \notin K$ and $bm \notin N$. Then $ab \in (N \cap K : M)$ is obtained similarly to Case 3.

Therefore the intersection of each pair of distinct prime subsemimodules of M is a 2-absorbing subsemimodule of M.

(ii) Let P and Q be two distinct weakly prime subsemimodules of M. Then $P \cap Q$ is a proper subsemimodule of M. Assume that $a, b \in R$ and $m \in M$ with $0 \neq abm \in P \cap Q$, but $am \notin P \cap Q$ and $bm \notin P \cap Q$. Then $0 \neq abm \in P$ and $0 \neq abm \in Q$. We claim that $ab \in (P \cap Q : M)$.

Case 1: $am \notin P$ and $bm \notin P$. Since $0 \neq abm \in P$ and P is a weakly prime subsemimodule, $bm \in P$ or $a \in (P : M)$. So $a \in (P : M)$ because $bm \notin P$. Thus $aM \subseteq P$ so that $am \in P$ contradicts $am \notin P$. Hence Case 1 is impossible.

Case 2: $am \notin Q$ and $bm \notin Q$. This is not possible either.

Case 3: $am \notin P$ and $bm \notin Q$. Since $0 \neq abm \in P$ which is a weakly prime subsemimodule, $am \in P$ or $b \in (P : M)$. Thus $b \in (P : M)$ because $am \notin P$. We obtain that $bM \subseteq P$. Then $abM = baM \subseteq bM \subseteq P$. Thus $ab \in (P : M)$. Similarly, we obtain that $ab \in (Q : M)$. Now, we have $ab \in (P : M)$ and $ab \in (Q : M)$. So $ab \in (P : M) \cap (Q : M) = (P \cap Q : M)$.

Case 4: $am \notin Q$ and $bm \notin P$. Again $ab \in (P \cap Q : M)$ similarly to Case 3.

Therefore the intersection of each pair of distinct weakly prime subsemimodules of M is a weakly 2-absorbing subsemimodule of M. □

However, it is not necessary true that the intersection of any finite (weakly) prime subsemimodules of M is a (weakly) 2-absorbing subsemimodule of M. For example, let $R = \mathbb{Z}_0^+$, M= $\mathbb{Z}_0^+$, N=$2\mathbb{Z}_0^+$, K=$3\mathbb{Z}_0^+$ and P=$5\mathbb{Z}_0^+$. Then N, K and P are prime subsemimodules of M. But $N \cap K \cap P$= $30\mathbb{Z}_0^+$ which is not a 2-absorbing subsemimodule of M because $3 \cdot 5 \cdot (2) \in 30\mathbb{Z}_0^+$ but $3 \cdot (2) \notin 30\mathbb{Z}_0^+$, $5 \cdot (2) \notin 30\mathbb{Z}_0^+$ and $3 \cdot 5 \notin (30\mathbb{Z}_0^+ : \mathbb{Z}_0^+)$.

Recall that if N is a Q-subsemimodule of an R-semimodule M, then we can construct a factor semimodule $M/N_{(Q)}$. The next results concern relationship between (weakly) 2-absorbing subsemimodules and (weakly) 2-absorbing Q-subsemimodules.

Theorem 3.2. *Let M be an R-semimodule, N a Q-subsemimodule of M and P a subtractive subsemimodule of M with $N \subseteq P$. Then P is a 2-absorbing subsemimodule of M if and only if $P/N_{(Q\cap P)}$ is a 2-absorbing subsemimodule of $M/N_{(Q)}$.*

Proof. First, assume that P is a 2-absorbing subsemimodule of M. Recall that $P/N_{(Q\cap P)}$ is a subsemimodule of $M/N_{(Q)}$ by Proposition 2.26. Moreover, $P/N_{(Q\cap P)}$ is proper because P is proper. Let $a, b \in R$ and $q_1 + N \in M/N_{(Q)}$, where $q_1 \in Q$, be such that $ab \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Then there exists unique $q_2 \in Q \cap P$ such that $ab \odot (q_1 + N) = q_2 + N$ where $abq_1 + N \subseteq q_2 + N$. Since $q_2 + N \subseteq P$, it follows that $abq_1 + N \subseteq P$. Since $N \subseteq P$ and P is a subtractive subsemimodule, $abq_1 \in P$. Since P is a 2-absorbing subsemimodule of M, it can be concluded that $aq_1 \in P$ or $bq_1 \in P$ or $abM \subseteq P$. We claim that $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $ab \in (P/N_{(Q\cap P)} : M/N_{(Q)})$.

Case 1: $aq_1 \in P$**.** Since $q_1 \in Q$, we have $aq_1 \in Q$. Then $aq_1 \in Q \cap P$, so $aq_1 + N \in P/N_{(Q\cap P)}$. Moreover, $a \odot (q_1 + N) = q' + N$ where $q' \in Q$ is unique such that $aq_1 + N \subseteq q' + N$. Then $(aq_1 + N) \cap (q' + N) \neq \varnothing$ so that $q' = aq_1 \in Q \cap P$. Thus $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$.

Case 2: $bq_1 \in P$**.** We can conclude similarly to Case 1 that $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$.

Case 3: $abM \subseteq P$**.** Let $q + N \in M/N_{(Q)}$ where $q \in Q$ and $ab \odot (q + N) = q_3 + N$

where $q_3 \in Q$ is unique such that $abq+N \subseteq q_3+N$. Then $abq+N = q_3+N$ since $abq \in Q$. Thus $q_3+N = abq+N \subseteq P$ since $abM \subseteq P$ and $N \subseteq P$. Hence $q_3 \in P$ because P is subtractive. As a result, $q_3 \in Q \cap P$. Then $ab \odot (q+N) = q_3+N \in P/N_{(Q\cap P)}$. Thus $ab \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$. Hence $ab \in (P/N_{(Q\cap P)} : M/N_{(Q)})$.

Therefore $P/N_{(Q\cap P)}$ is a 2-absorbing subsemimodule of $M/N_{(Q)}$.

Conversely, assume that $P/N_{(Q\cap P)}$ is a 2-absorbing subsemimodule of $M/N_{(Q)}$. Then P is a proper subsemimodule of M. Let $a, b \in R$ and $m \in M$ be such that $abm \in P$. Then by Proposition 2.25, there is unique $q_1 \in Q$ such that $m \in q_1+N$ and $abm \in ab\odot(q_1+N)$. Let $ab\odot(q_1+N) = q_2+N$ where q_2 is the unique element of Q such that $abq_1+N \subseteq q_2+N$. Now, $abm \in P$ and $abm \in q_2+N$. So there is $n \in N$ such that $q_2+n = abm \in P$. Since P is subtractive and $n \in N \subseteq P$, we obtain $q_2 \in P$. Then $q_2 \in Q \cap P$. Thus $ab \odot (q_1+N) = q_2+N \in P/N_{(Q\cap P)}$. Since $P/N_{(Q\cap P)}$ is a 2-absorbing subsemimodule, $a \odot (q_1+N) \in P/N_{(Q\cap P)}$ or $b \odot (q_1+N) \in P/N_{(Q\cap P)}$ or $ab \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$.

Case 1: $a\odot(q_1+N) \in P/N_{(Q\cap P)}$. Then $a\odot(q_1+N) = q'+N$ where q' is the unique element of $Q\cap P$ such that $aq_1+N \subseteq q'+N$. Since $a \odot (q_1+N) = q'+N \subseteq P$, we get $aq_1+N \subseteq P$. Thus $aq_1 \in P$ because P is subtractive and $N \subseteq P$. Since $q_1 \in Q$, we have $aq_1 \in Q$. Then $aq_1 \in Q\cap P$. Since q' is the unique element of $Q\cap P$ such that $aq_1+N \subseteq q'+N$ and $aq_1 \in Q\cap P$, we obtain that $q' = aq_1$. It follows from $m \in q_1+N$ that $am \in a(q_1+N) \subseteq aq_1+N = q'+N = a\odot(q_1+N) \subseteq P$. Thus $am \in P$.

Case 2: $b \odot (q_1+N) \in P/N_{(Q\cap P)}$. Again, this is similar to Case 1, so we can conclude that $bm \in P$.

Case 3: $ab \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$. Let $x \in M$. By Proposition 2.25, there is unique $q_3 \in Q$ such that $x \in q_3+N$ and $abx \in ab \odot (q_3+N) = q_4+N$ where q_4 is the unique element of Q such that $abq_3+N \subseteq q_4+N$. Now, $q_4+N = ab \odot (q_3+N) \in P/N_{(Q\cap P)}$. Then $abx \in q_4+N \subseteq P$. Thus $abM \subseteq P$.

Therefore P is a 2-absorbing subsemimodule of M. □

Theorem 3.3. *Let M be an R-semimodule, N a Q-subsemimodule of M and P a subtractive subsemimodule of M with $N \subseteq P$.*

(i) If P is a weakly 2-absorbing subsemimodule of M, then $P/N_{(Q\cap P)}$ is a weakly 2-absorbing subsemimodule of $M/N_{(Q)}$.

(ii) If N and $P/N_{(Q\cap P)}$ are weakly 2-absorbing subsemimodules of M and $M/N_{(Q)}$, respectively, then P is a weakly 2-absorbing subsemimodule of M.

Proof. (i) Assume that P is a weakly 2-absorbing subsemimodule of M. Then $P/N_{(Q\cap P)}$ is a proper subsemimodule of $M/N_{(Q)}$. Let $a, b \in R$ and $q_1 + N \in M/N_{(Q)}$, where $q_1 \in Q$, be such that $0_M + N \neq ab \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Then there exists unique $q_2 \in Q \cap P$ such that $ab \odot (q_1 + N) = q_2 + N$ where $abq_1 + N \subseteq q_2 + N$. Since $q_2 + N \subseteq P$, it follows that $abq_1 + N \subseteq P$. Since $N \subseteq P$ and P is a subtractive subsemimodule, $abq_1 \in P$.

Case 1: $abq_1 = 0$. Since $abq_1 \in (0_M + N) \cap (q_2 + N)$, we obtain that $0_M = q_2$. Thus, $0_M + N = q_2 + N$ contradicts the fact that $q_2 + N = ab \odot (q_1 + N) \neq 0_M + N$. This case is absurd.

Case 2: $abq_1 \neq 0$. Since P is a weakly 2-absorbing subsemimodule of M, it can be concluded that $aq_1 \in P$ or $bq_1 \in P$ or $abM \subseteq P$. We claim that $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $ab \in (P/N_{(Q\cap P)} : M/N_{(Q)})$.

Subcase 2.1: $aq_1 \in P$**.** Then $aq_1 \in Q \cap P$, so $aq_1 + N \in P/N_{(Q\cap P)}$. Moreover, $a \odot (q_1 + N) = q' + N$ where $q' \in Q$ is unique such that $aq_1 + N \subseteq q' + N$. Then $(aq_1 + N) \cap (q' + N) \neq \varnothing$ so that $q' = aq_1 \in Q \cap P$. Thus $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$.

Subcase 2.2: $bq_1 \in P$**.** We can conclude similarly to Subcase 2.1 that $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$.

Subcase 2.3: $abM \subseteq P$**.** Let $q + N \in M/N_{(Q)}$ where $q \in Q$. Let $ab \odot (q + N) = q_3 + N$ where $q_3 \in Q$ is unique such that $abq + N \subseteq q_3 + N$. Then $abq + N = q_3 + N$ since $abq \in Q$. Then $q_3 + N = abq + N \subseteq P$ since $abM \subseteq P$ and $N \subseteq P$ so that $q_3 \in P$ because P is subtractive. Thus $q_3 \in Q \cap P$. Then $ab \odot (q + N) = q_3 + N \in P/N_{(Q\cap P)}$. Thus $ab \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$. Hence $ab \in (P/N_{(Q\cap P)} : M/N_{(Q)})$.

Therefore $P/N_{(Q\cap P)}$ is a weakly 2-absorbing subsemimodule of $M/N_{(Q)}$.

(ii) Assume that N and $P/N_{(Q\cap P)}$ are weakly 2-absorbing subsemimodules of M and $M/N_{(Q)}$, respectively. Then P is a proper subsemimodule of M. Let $0 \neq abm \in P$ where $a, b \in R$ and $m \in M$.

Case 1: $0 \neq abm \in N$. Then $am \in N \subseteq P$ or $bm \in N \subseteq P$ or $ab \in (N : M) \subseteq (P : M)$.

Case 2: $0 \neq abm \in P \setminus N$. Then by Proposition 2.25, there is unique $q_1 \in Q$ such that $m \in q_1 + N$ and $abm \in ab \odot (q_1 + N)$. Let $ab \odot (q_1 + N) = q_2 + N$ where q_2 is a unique element of Q such that $abq_1 + N \subseteq q_2 + N$. Now, $abm \in P$ and $abm \in q_2 + N$. So there is $n \in N$ such that $q_2 + n = abm \in P$. Since P is subtractive and $n \in N \subseteq P$, we obtain $q_2 \in P$. Then $q_2 \in Q \cap P$. Suppose that $0_M + N = ab \odot (q_1 + N)$. Since $q_2 + N = ab \odot (q_1 + N) = 0_M + N$ and $abm \in q_2 + N$, it follows that $abm \in 0_M + N = N$ contradicts the fact that $abm \in P \setminus N$. Thus $0_M + N \neq ab \odot (q_1 + N) = q_2 + N \in P/N_{(Q\cap P)}$. Since $P/N_{(Q\cap P)}$ is a weakly 2-absorbing subsemimodule, $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $ab \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$.

Subcase 2.1: $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Then $a \odot (q_1 + N) = q' + N$ where q' is a unique element of $Q \cap P$ such that $aq_1 + N \subseteq q' + N$. Since $a \odot (q_1 + N) = q' + N$ and $a \odot (q_1 + N) \subseteq P$, we get $aq_1 + N \subseteq P$. Thus $aq_1 \in P$ because P is subtractive and $N \subseteq P$. Then $aq_1 \in Q \cap P$. So $q' = aq_1$. Since $m \in q_1 + N$, it follows that $am \in a(q_1 + N) \subseteq aq_1 + N = q' + N = a \odot (q_1 + N) \subseteq P$. Thus $am \in P$.

Subcase 2.2: $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Again, this is similar to Subcase 2.1, we can conclude that $bm \in P$.

Subcase 2.3: $abM/N_{(Q)} \subseteq P/N_{(Q\cap P)}$. Let $x \in M$. By Proposition 2.25, there is unique $q_3 \in Q$ such that $x \in q_3 + N$ and $abx \in ab \odot (q_3 + N) = q_4 + N$ where q_4 is a unique element of Q such that $abq_3 + N \subseteq q_4 + N$. Now, $q_4 + N = ab \odot (q_3 + N) \in P/N_{(Q\cap P)}$. Then $abx \in q_4 + N \subseteq P$. Thus $abM \subseteq P$.

Therefore P is a weakly 2-absorbing subsemimodule of M. □

We observe from Theorem 3.2 that P is a 2-absorbing subsemimodule of M if and only if $P/N_{(Q\cap P)}$ is a 2-absorbing subsemimodule of $M/N_{(Q)}$. However, this is not true for the weakly 2-absorbing subsemimodule. Theorem 3.3 points out that if P is a weakly 2-absorbing subsemimodule of M, then $P/N_{(Q\cap P)}$ is a weakly 2-absorbing subsemimodule of $M/N_{(Q)}$ but not vice versa. Consider the proof of Theorem 3.3 (ii). If $0 \neq abm \in P$ where $a, b \in R$ and $m \in M$, then it is not sufficient to ensure that $0_M + N \neq ab \odot (q_1 + N)$.

Recall that 2-absorbing subsemimodules are weakly 2-absorbing subsemimodules. But its converse does not necessary hold. Therefore some conditions are needed in order to make the converse true.

Theorem 3.4. *Let M be an R-semimodule and N a weakly 2-absorbing subsemimodule of M. If N is a subtractive subsemimodule and $(N : M)^2N \neq \{0\}$, then N is a 2-absorbing subsemimodule.*

Proof. Assume that N is a subtractive subsemimodule and $(N : M)^2N \neq \{0\}$. Proposition 2.13 provides that $(N : M)$ is a subtractive ideal of R. Let $a, b \in R$ and $m \in M$ be such that $abm \in N$. We claim that $am \in N$ or $bm \in N$ or $ab \in (N : M)$. If $0 \neq abm \in N$, then we are through because N is a weakly 2-absorbing subsemimodule of M. Then assume that $abm = 0$.

Case 1: $abN \neq \{0\}$. Then there is $n_0 \in N$ such that $abn_0 \neq 0$. Now $0 \neq abn_0 = 0 + abn_0 = abm + abn_0 \in N$ and $ab(m + n_0) = abm + abn_0$. Since N is weakly 2-absorbing, we obtain that $am+an_0 = a(m+n_0) \in N$ or $bm+bn_0 = b(m+n_0) \in N$ or $ab \in (N : M)$. Since N is a subtractive subsemimodule and $an_0, bn_0 \in N$, it follows that $am \in N$ or $bm \in N$ or $ab \in (N : M)$.

Case 2: $abN = \{0\}$. Consider the choices of $a(N : M)M$ and $b(N : M)M$.

Subcase 2.1: $a(N : M)M \neq \{0\}$ or $b(N : M)M \neq \{0\}$. Without loss of generality, we assume that $a(N : M)M \neq \{0\}$. Then there exists $r \in (N : M)$ such that $arm \neq 0$. Thus $0 \neq arm = abm + arm = a(b + r)m \in N$. Since N is weakly 2-absorbing, $am \in N$ or $(b + r)m \in N$ or $a(b + r) \in (N : M)$. If $(b + r)m \in N$ or $a(b + r) \in (N : M)$, then applying the fact that N and $(N : M)$ are subtractive leads to $bm \in N$ or $ab \in (N : M)$. Thus $am \in N$ or $bm \in N$ or $ab \in (N : M)$.

Subcase 2.2: $a(N:M)M=\{0\}$ and $b(N:M)M=\{0\}$. Since $(N:M)^2N\neq\{0\}$, there exist $a_0,b_0\in(N:M)$ and $x_0\in N$ with $0\neq a_0b_0x_0\in N$. Then $a_0b_0m\in N$. Since $a(N:M)M=\{0\}$ and $b(N:M)M=\{0\}$, we obtain that $ab_0m=0$, $a_0bm=0$, $ab_0x_0=0$ and $a_0bx_0=0$. In addition, $abx_0=0$ because $abN=\{0\}$.

Subcase 2.2.1: $a_0b_0m\neq 0$. Then $0\neq a_0b_0m=abm+ab_0m+a_0bm+a_0b_0m=(a+a_0)(b+b_0)m$. Besides, $(a+a_0)(b+b_0)m\in N$ because $a_0b_0m\in N$. Since N is weakly 2-absorbing and $0\neq(a+a_0)(b+b_0)m\in N$, we obtain that $(a+a_0)m\in N$ or $(b+b_0)m\in N$ or $(a+a_0)(b+b_0)\in(N:M)$. Thus $am+a_0m\in N$ or $bm+b_0m\in N$ or $ab+ab_0+a_0b+a_0b_0\in(N:M)$. Since $a_0,b_0\in(N:M)$ which is an ideal, it follows that $a_0m,b_0m\in N$ and $ab_0+a_0b+a_0b_0\in(N:M)$. Being subtractive of N and $(N:M)$ implies that $am\in N$ or $bm\in N$ or $ab\in(N:M)$.

Subcase 2.2.2: $a_0b_0m=0$. Then $(a+a_0)(b+b_0)(m+x_0)=abm+abx_0+ab_0m+ab_0x_0+a_0bm+a_0bx_0+a_0b_0m+a_0b_0x_0=a_0b_0x_0\neq 0$. We obtain that $0\neq(a+a_0)(b+b_0)(m+x_0)\in N$ because $0\neq a_0b_0x_0\in N$. Since N is weakly 2-absorbing, $(a+a_0)(m+x_0)\in N$ or $(b+b_0)(m+x_0)\in N$ or $(a+a_0)(b+b_0)\in(N:M)$.

Subcase 2.2.2.1: $(a+a_0)(m+x_0)\in N$. Then $am+ax_0+a_0m+a_0x_0=(a+a_0)(m+x_0)\in N$. Since N is subtractive and $ax_0,a_0m,a_0x_0\in N$, we conclude that $am\in N$.

Subcase 2.2.2.2: $(b+b_0)(m+x_0)\in N$. Similarly to Subcase 2.2.2.1, we obtain that $bm\in N$.

Subcase 2.2.2.3: $(a+a_0)(b+b_0)\in(N:M)$. Then $ab+ab_0+a_0b+a_0b_0=(a+a_0)(b+b_0)\in(N:M)$. We know that $a_0b,ab_0,a_0b_0\in(N:M)$ which is subtractive, $ab\in(N:M)$.

Therefore N is a 2-absorbing subsemimodule. □

Corollary 3.5. *Let M be an R-semimodule and N a subtractive subsemimodule of M. If N is weakly 2-absorbing but not 2-absorbing, then the followings hold.*

(i) N is nilpotent.

(ii) If M is a multiplication R-semimodule, then $N^3=\{0\}$.

Proof. Assume that N is weakly 2-absorbing but not 2-absorbing.

(i) Theorem 3.4 yields that $(N : M)^2N = \{0\}$ which can be concluded that N is nilpotent.

(ii) Assume that M is a multiplication R-semimodule. Then $N = (N : M)M$. Consequently, $N^3 = (N : M)^3M = (N : M)^2(N : M)M = (N : M)^2N = \{0\}$ by Theorem 3.4. □

Next result, we provide a condition that weakly 2-absorbing subsemimodules are 2-absorbing subsemimodules. However, this can be done in the case of faithful multiplication R-semimodules. Recall that for an R-semimodule M, $\mathrm{rad}_M(\{0\})$ is the intersection of all prime subsemimodules of M.

Proposition 3.6. *Let M be a faithful multiplication R-semimodule and N a weakly 2-absorbing subtractive subsemimodule of M. If N is not 2-absorbing, then $N \subseteq \mathrm{rad}_M(\{0\})$.*

Proof. Assume that N is not a 2-absorbing subsemimodule. Then $(N : M)^2N = \{0\}$ by Theorem 3.4. We claim that $(N : M)^3 \subseteq \big((N : M)^2N : M\big)$. To show this, let $r \notin \big((N : M)^2N : M\big)$. Then there exists $m \in M$ such that $rm \notin (N : M)^2N$. Since M is a multiplication R-semimodule, $N = (N : M)M$. Thus $rm \notin (N : M)^2(N : M)M = (N : M)^3M$. Then $r \notin (N : M)^3$. Therefore $(N : M)^3 \subseteq \big((N : M)^2N : M\big)$ as claimed. We have $\{0\} \subseteq (N : M)^3 \subseteq \big((N : M)^2N : M\big) = (\{0\} : M) = \{0\}$ since M is faithful. Thus $(N : M)^3 = \{0\}$. Now, we have $(N : M) \subseteq \sqrt{\{0\}}$. Then $N = (N : M)M \subseteq \sqrt{\{0\}}M = \sqrt{(\{0\} : M)}M \subseteq \mathrm{rad}_M(\{0\})$ because of Proposition 2.38.

Therefore $N \subseteq \mathrm{rad}_M(\{0\})$. □

Proposition 3.6 shows that for a weakly 2-absorbing subtractive subsemimodule N of a faithful multiplication R-semimodule M, if N is not contained in the intersection of all prime subsemimodules of M, then N must be a 2-absorbing subsemimodule.

We found that being cyclic of an R-semimodule plays a major tool for our results. Recall that cyclic R-semimodules are multiplication R-semimodules by Proposition 2.31.

We show in the followings that there are some relationships between being (weakly) 2-absorbing subsemimodules of N and being (weakly) 2-absorbing ideals of $(N : M)$ where N is a subsemimodule of a cyclic R-semimodule M.

Proposition 3.7. *Let M be a cyclic R-semimodule and N a subsemimodule of M. Then the followings hold.*

(i) N is a 2-absorbing subsemimodule of M if and only if $(N : M)$ is a 2-absorbing ideal of R.

(ii) If, in addition, M is faithful, then N is a weakly 2-absorbing subsemimodule of M if and only if $(N : M)$ is a weakly 2-absorbing ideal of R.

Proof. Let $M = Rm$ for some $m \in M$. Proposition 2.29 yields $(N : M) = (N : m)$. Note that N is a proper subsemimodule if and only if $(N : M)$ is a proper ideal.

(i) First, assume that N is a 2-absorbing subsemimodule of M. Let $a, b, c \in R$ be such that $abc \in (N : M)$ but $ab \notin (N : M)$ and $ac \notin (N : M)$. Then there exist $r, s \in R$ such that $ab(rm) \notin N$ and $ac(sm) \notin N$. Thus $abm \notin N$ and $acm \notin N$. Since $abc \in (N : M)$ and $m \in M$, we get $abcm \in N$. Then $bc(am) \in N$, $a(bm) \notin N$ and $c(am) \notin N$ since R is commutative. Thus $bc \in (N : M)$ because N is 2-absorbing.

Therefore $(N : M)$ is a 2-absorbing ideal of R.

Conversely, assume that $(N : M)$ is a 2-absorbing ideal of R. Let $a, b \in R$ and $x \in M$ be such that $abx \in N$. Then there exists $c \in R$ such that $x = cm$, so $abcm \in N$, i.e., $abc \in (N : m) = (N : M)$. Since $(N : M)$ is a 2-absorbing ideal and $(N : m) = (N : M)$, we obtain that $ac \in (N : m)$ or $bc \in (N : m)$ or $ab \in (N : M)$. Therefore $ax = acm \in N$ or $bx = bcm \in N$ or $ab \in (N : M)$.

Therefore N is a 2-absorbing subsemimodule of M.

(ii) Assume further that M is faithful.

First, let N be a weakly 2-absorbing subsemimodule of M. Let $a, b, c \in R$ be such that $0 \neq abc \in (N : M)$ but $ab \notin (N : M)$ and $ac \notin (N : M)$. Then there exist $r, s \in R$ such that $ab(rm) \notin N$ and $ac(sm) \notin N$. Thus $abm \notin N$ and $acm \notin N$. Suppose that $abcm = 0$. So, $\{0\} = abcRm = abcM$. Since M is faithful, $abc = 0$ which is a contradiction. Thus $abcm \neq 0$. Now, $0 \neq abcm \in N$. Since R is commutative, $0 \neq bc(am) \in N$. Since N is a weakly 2-absorbing subsemimodule, $b(am) \notin N$ and $c(am) \notin N$, we obtain that $bc \in (N : M)$.

Therefore $(N : M)$ is a weakly 2-absorbing ideal of R.

Next, let $(N : M)$ be a weakly 2-absorbing ideal of R. Let $a, b \in R$ and $x \in M$ be such that $0 \neq abx \in N$. Then there exists $c \in R$ such that $x = cm$, so $0 \neq abcm \in N$. Thus $0 \neq abc \in (N : m)$. Since $(N : M)$ is a weakly 2-absorbing ideal and $(N : m) = (N : M)$, we obtain that $ac \in (N : m)$ or $bc \in (N : m)$ or $ab \in (N : M)$. Hence $ax = acm \in N$ or $bx = bcm \in N$ or $ab \in (N : M)$.

Therefore N is a weakly 2-absorbing subsemimodule of M. □

In fact, for a subsemimodule N of a cyclic R-semimodule M, if the ideal $(N : M)$ is weakly 2-absorbing, then the subsemimodule N is also weakly 2-absorbing without the requirement that M has to be faithful.

In a commutative semiring R, if a proper ideal I of R is also subtractive, Darani showed in [11] the equivalent definition of being 2-absorbing ideal of I as follows:

> I is a 2-absorbing ideal of R if and only if $I_1I_2I_3 \subseteq I$ implies that $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$ for any ideals I_1, I_2 and I_3 of R.

We also obtain the equivalent definition of being 2-absorbing subsemimodules in the similar fashion. However, R-semimodules need to be cyclic R-semimodules.

Proposition 3.8. *Let M be an R-semimodule and N a proper subsemimodule of M satisfying the following property: for any ideals I and J of R and a subsemimodule P of M, if $IJP \subseteq N$, then $IP \subseteq N$ or $JP \subseteq N$ or $IJ \subseteq (N : M)$. Then N is a 2-absorbing subsemimodule.*

Proof. Let $a, b \in R$ and $m \in M$ be such that $abm \in N$. Recall that Ra and Rb are ideals of R and Rm is a subsemimodule of M. Therefore $RaRbRm \subseteq N$. By assumption, we get $RaRm \subseteq N$ or $RbRm \subseteq N$ or $RaRb \subseteq (N : M)$. Since $1 \in R$, it follows that $am \in N$ or $bm \in N$ or $ab \in (N : M)$.

Therefore N is a 2-absorbing subsemimodule. □

Proposition 3.9. *Let M be a cyclic R-semimodule and N a 2-absorbing subtractive subsemimodule of M. Then, for any ideals I and J of R and a subsemimodule P of M, if $IJP \subseteq N$, then $IP \subseteq N$ or $JP \subseteq N$ or $IJ \subseteq (N : M)$.*

Proof. Note that $(N : M)$ is a 2-absorbing subtractive ideal of R by Proposition 2.13 and Proposition 3.7. Let I and J be ideals of R and P a subsemimodule of M such that $IJP \subseteq N$. Since M is cyclic, M is a multiplication R-semimodule. Then there exists an ideal I' of R such that $P = I'M$. Thus $IJI'M \subseteq N$, i.e. $IJI' \subseteq (N : M)$. Since $(N : M)$ is a 2-absorbing subtractive ideal of R, we obtain that $IJ \subseteq (N : M)$ or $II' \subseteq (N : M)$ or $JI' \subseteq (N : M)$. That is $IJ \subseteq (N : M)$ or $II'M \subseteq N$ or $JI'M \subseteq N$.

Therefore $IJ \subseteq (N : M)$ or $IP \subseteq N$ or $JP \subseteq N$. □

Being cyclic of the R-semimodule M in Proposition 3.9 is necessary eventhough in its proof, it seems that having M be multiplication should be enough. This requirement is needed because of making use of Proposition 3.7.

Proposition 3.10. *Let M be a cyclic R-semimodule and N a proper subtractive subsemimodule of M. The following statements are equivalent.*

(i) N is a 2-absorbing subsemimodule.

(ii) For any ideals I and J of R and a subsemimodule P of M, if $IJP \subseteq N$, then $IP \subseteq N$ or $JP \subseteq N$ or $IJ \subseteq (N : M)$.

Proof. The proof follows from Proposition 3.8 and Proposition 3.9. □

A subsemimodule N satisfying (ii) in Proposition 3.10 is called a ***strongly 2-absorbing subsemimodule***.

Corollary 3.11. *Let M be a cyclic R-semimodule and N a proper subtractive subsemimodule of M. If N is a 2-absorbing subsemimodule of M, then for any subsemimodules U, V, W of M such that $UVW \subseteq N$ implies $UV \subseteq N$ or $UW \subseteq N$ or $VW \subseteq N$.*

Proof. Let U, V and W be subsemimodules of M be such that $UVW \subseteq N$. Since M is cyclic, M is a multiplication R-semimodule. Then there exist ideals I, J and K of R such that $U = IM$, $V = JM$ and $W = KM$. Thus $IJ(KM) = (IM)(JM)(KM) = UVW \subseteq N$. By Proposition 3.9, we obtain that $I(KM) \subseteq N$ or $J(KM) \subseteq N$ or $IJ \subseteq (N : M)$. Thus $IMKM \subseteq N$ or $JMKM \subseteq N$ or $IMJM \subseteq N$.

Therefore $UW \subseteq N$ or $VW \subseteq N$ or $UV \subseteq N$. □

Nevertheless, the converse of Corollary 3.11 does not necessarily hold. For example, consider where $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_{12}$ and $N = \{\bar{0}\}$. All subsemimodules of $\mathbb{Z}_{12}$ are known to be $\{\bar{0}\}, \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}, \{\bar{0}, \bar{4}, \bar{8}\}, \{\bar{0}, \bar{6}\}, \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$ and $\mathbb{Z}_{12}$. We claim that for any subsemimodules U, V and W of $\mathbb{Z}_{12}$ with $UVW \subseteq \{\bar{0}\}$ implies $UV \subseteq \{\bar{0}\}$ or $UW \subseteq \{\bar{0}\}$ or $VW \subseteq \{\bar{0}\}$. It is clear that $\{\bar{0}, \bar{6}\}\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}\}\mathbb{Z}_{12} \subseteq \{\bar{0}\}$ implies $\{\bar{0}, \bar{6}\}\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\} \subseteq \{\bar{0}\}$. By the same manner, we obtain that any subsemimodules U, V and W of $\mathbb{Z}_{12}$ with $UVW \subseteq \{\bar{0}\}$ implies $UV \subseteq \{\bar{0}\}$ or $UW \subseteq \{\bar{0}\}$ or $VW \subseteq \{\bar{0}\}$. This shows that the converse of Corollary 3.11 does not hold because $\{\bar{0}\}$ is not a 2-absorbing subsemimodule of $\mathbb{Z}_{12}$ by Proposition 2.22.

Recall that $m_1m_2m_3$ is the product of the subsemimodules Rm_1, Rm_2 and Rm_3, for any $m_1, m_2, m_3 \in M$, i.e., $m_1m_2m_3 = Rm_1Rm_2Rm_3 = (I_1M)(I_2M)(I_3M) = (I_1I_2I_3)M$ where I_1, I_2 and I_3 are presentation ideals of Rm_1, Rm_2 and Rm_3, respectively.

Corollary 3.12. *Let M be a cyclic R-semimodule and N a proper subtractive subsemimodule of M. If N is a 2-absorbing subsemimodule of M, then for any $m_1, m_2, m_3 \in M$ such that $m_1m_2m_3 \subseteq N$ implies $m_1m_2 \subseteq N$ or $m_1m_3 \subseteq N$ or $m_2m_3 \subseteq N$.*

Proof. Assume that N is a 2-absorbing subsemimodule of M. Let $m_1, m_2, m_3 \in M$ be such that $m_1m_2m_3 \subseteq N$. Since Rm_1, Rm_2 and Rm_3 are subsemimodules of M and $Rm_1Rm_2Rm_3 = m_1m_2m_3 \subseteq N$, we obtain that $Rm_1Rm_2 \subseteq N$ or $Rm_1Rm_3 \subseteq N$ or $Rm_2Rm_3 \subseteq N$ by Corollary 3.11.

Therefore $m_1m_2 \subseteq N$ or $m_1m_3 \subseteq N$ or $m_2m_3 \subseteq N$. □

If the condition that $m_1m_2m_3 \subseteq N$ implies $m_1m_2 \subseteq N$ or $m_1m_3 \subseteq N$ or $m_2m_3 \subseteq N$ in Corollary 3.12 is slightly changed, then the new condition forces N to be a 2-absorbing subsemimodule.

Proposition 3.13. *Let M be a cyclic R-semimodule and N a proper subtractive subsemimodule of M. If for any $m_1, m_2, m_3 \in M$ such that $m_1m_2m_3 \subseteq N$ implies $m_1 \in N$ or $m_2 \in N$ or $m_3 \in N$, then N is a 2-absorbing subsemimodule of M.*

Proof. Assume that for any $m_1, m_2, m_3 \in M$ such that $m_1m_2m_3 \subseteq N$ implies $m_1 \in N$ or $m_2 \in N$ or $m_3 \in N$. Suppose that N is not a 2-absorbing subsemimodule. By Proposition 3.10, let I and J be ideals of R and P a subsemimodule of M such that $IJP \subseteq N$ but $IP \nsubseteq N$, $JP \nsubseteq N$ and $IJ \nsubseteq (N : M)$. Then there exist $p \in IP \setminus N$, $p' \in JP \setminus N$, $a \in IJ \setminus (N : M)$ and $m \in M$ such that $am \notin N$. Since M is cyclic, M is a multiplication R-semimodule. Let $P = I'M$ where I' is an ideal of R. Then $pp'am = RpRp'Ram \subseteq (RIP)(RJP)(RIJM) \subseteq (IP)(JP)(IJM) = (II'M)(JI'M)(IJM) = (II'JI'IJ)M \subseteq (IJI')M = IJ(I'M) = IJP \subseteq N$. By assumption, it follows that $p \in N$ or $p' \in N$ or $am \in N$ contradicts the fact that $p \in IP \setminus N$, $p' \in JP \setminus N$ and $am \notin N$.

Therefore N is a 2-absorbing subsemimodule of M. □

Proposition 3.14. *Let M be a cyclic R-semimodule and N a proper subtractive subsemimodule of M. If for any subsemimodules U, V and W of M such that $UVW \subseteq N$ implies $U \subseteq N$ or $V \subseteq N$ or $W \subseteq N$, then N is a 2-absorbing subsemimodule of M.*

Proof. Assume that N is not a 2-absorbing subsemimodule. By Proposition 3.13, there are $m_1, m_2, m_3 \in M$ with $m_1m_2m_3 \subseteq N$ but $m_1 \notin N$, $m_2 \notin N$ and $m_3 \notin N$.

Then $Rm_1 \nsubseteq N$, $Rm_2 \nsubseteq N$ and $Rm_3 \nsubseteq N$. Note that Rm_1, Rm_2 and Rm_3 are subsemimodules of M and $Rm_1Rm_2Rm_3 = m_1m_2m_3 \subseteq N$. Then the proof is complete. □

The converse of Proposition 3.13 and Proposition 3.14 does not necessary hold. For example, consider where $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_6$ and $N = \{\bar{0}\}$ which is a 2-absorbing subtractive subsemimodule of $\mathbb{Z}_6$. We know that $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ are subsemimodules of M and $\{\bar{0}, \bar{3}\}\,\{\bar{0}, \bar{3}\}\,\{\bar{0}, \bar{2}, \bar{4}\} \subseteq N$. But $\{\bar{0}, \bar{3}\} \nsubseteq N$ and $\{\bar{0}, \bar{2}, \bar{4}\} \nsubseteq N$.

We end this chapter by studying the collection of all 2-absorbing subtractive subsemimodules of an R-semimodule. It turns out that this collection does satisfy the Zariski topology. Anyhow, we first suggest some definitions and notation.

Definition 3.15. A subsemimodule L of an R-semimodule M is said to be ***semi-2-absorbing*** if L is an intersection of 2-absorbing subtractive subsemimodules of M.

We see that 2-absorbing subtractive subsemimodules are semi-2-absorbing subsemimodules. But the converse does not necessary hold. For example, let $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_0^+$, $N = 2\mathbb{Z}_0^+$, $L = 3\mathbb{Z}_0^+$ and $P = 5\mathbb{Z}_0^+$. Thus N, L and P are 2-absorbing subtractive subsemimodules of M so that $N \cap L \cap P = 30\mathbb{Z}_0^+$ is semi-2-absorbing. But $N \cap L \cap P = 30\mathbb{Z}_0^+$ is not a 2-absorbing subtractive subsemimodule of M because $3 \cdot 5 \cdot (2) \in 30\mathbb{Z}_0^+$ but $3 \cdot (2) \notin 30\mathbb{Z}_0^+$, $5 \cdot (2) \notin 30\mathbb{Z}_0^+$ and $3 \cdot 5 \notin (30\mathbb{Z}_0^+ : \mathbb{Z}_0^+)$.

Notation. For a subsemimodule N of an R-semimodule M, let

$\overline{\mathrm{rad}_M(N)}$ be the intersection of all 2-absorbing subtractive subsemimodules of M containing N,

$V'(N)$ be the collection of all 2-absorbing subtractive subsemimodules of M containing N, and

$\overline{\mathrm{spec}(M)}$ be the collection of all 2-absorbing subtractive subsemimodules of M.

Clearly, $V'(M) = \varnothing$, $V'(\{0\}) = \overline{\mathrm{spec}(M)}$ and $N \subseteq \overline{\mathrm{rad}_M(N)}$.

Definition 3.16. A 2-absorbing subtractive subsemimodule N of an R-semimodule M is called ***extraordinary*** if whenever A and B are semi-2-absorbing subtractive subsemimodules of M with $A \cap B \subseteq N$, then $A \subseteq N$ or $B \subseteq N$.

Example. Let $R = \mathbb{Z}_0^+$, $M = \mathbb{Z}_0^+$ and $N = 3\mathbb{Z}_0^+$. Moreover, let A and B be semi-2-absorbing subtractive subsemimodules of M with $A \cap B \subseteq N$. Then $A = m\mathbb{Z}_0^+$, $B = m'\mathbb{Z}_0^+$ and $A \cap B = 3k\mathbb{Z}_0^+$ where $m, m', k \in \mathbb{Z}_0^+$. It can be shown that $A \cap B = \text{lcm}(m, m')\mathbb{Z}_0^+$. Thus $\text{lcm}(m, m')\mathbb{Z}_0^+ = 3k\mathbb{Z}_0^+$ so that $3 \mid m$ or $3 \mid m'$. Hence $A = m\mathbb{Z}_0^+ \subseteq 3\mathbb{Z}_0^+$ or $B = m'\mathbb{Z}_0^+ \subseteq 3\mathbb{Z}_0^+$. Therefore $N = 3\mathbb{Z}_0^+$ is extraordinary.

Proposition 3.17. *Let M be an R-semimodule. Then the following statements hold.*

(i) If P and L are subsemimodules of M such that $P \subseteq L$, then $V'(L) \subseteq V'(P)$.

(ii) If N is a subsemimodule of M, then $V'(N) = V'\left(\overline{\text{rad}_M(N)}\right)$.

(iii) If $\{N_i\}_{i\in I}$ is a family of subsemimodules of M, then $V'(\sum_{i\in I} N_i) = \bigcap_{i\in I} V'(N_i)$.

Proof. (i) Assume that P and L are subsemimodules of M such that $P \subseteq L$. Let $A \in V'(L)$. Thus A is a 2-absorbing subtractive subsemimodule and $L \subseteq A$. Since $P \subseteq L \subseteq A$, we obtain that $A \in V'(P)$.

Therefore $V'(L) \subseteq V'(P)$.

(ii) Assume that N is a subsemimodule of M. First, let $P \in V'(N)$. Thus P is a 2-absorbing subtractive subsemimodule of M and $N \subseteq P$. Then $\overline{\text{rad}_M(N)} \subseteq P$. That is $P \in V'\left(\overline{\text{rad}_M(N)}\right)$. Hence $V'(N) \subseteq V'\left(\overline{\text{rad}_M(N)}\right)$. Next, let $P \in V'\left(\overline{\text{rad}_M(N)}\right)$. Then $N \subseteq \overline{\text{rad}_M(N)} \subseteq P$. Thus $P \in V'(N)$. Hence $V'\left(\overline{\text{rad}_M(N)}\right) \subseteq V'(N)$.

Therefore $V'(N) = V'\left(\overline{\text{rad}_M(N)}\right)$.

(iii) Assume that $\{N_i\}_{i\in I}$ is a family of subsemimodules of M. First, let $P \in V'(\sum_{i\in I} N_i)$. Then $N_j \subseteq \sum_{i\in I} N_i \subseteq P$ for all $j \in I$. That is $P \in V'(N_j)$ for all $j \in I$.

Thus $P \in \bigcap_{i\in I} V'(N_i)$. Hence $V'(\sum_{i\in I} N_i) \subseteq \bigcap_{i\in I} V'(N_i)$. Next, let $P \in \bigcap_{i\in I} V'(N_i)$. Then $N_i \subseteq P$ for all $i \in I$. Thus $\sum_{i\in I} N_i \subseteq P$. That is $P \in V'(\sum_{i\in I} N_i)$. Hence $\bigcap_{i\in I} V'(N_i) \subseteq V'(\sum_{i\in I} N_i)$.

Therefore $V'(\sum_{i\in I} N_i) = \bigcap_{i\in I} V'(N_i)$. □

Let M be an R-semimodule and $\xi(M)$ denote the collection of all subsets $V'(N)$ of $\overline{\text{spec}(M)}$. Then $\xi(M)$ contains $\varnothing$ and $\overline{\text{spec}(M)}$; moreover, $\xi(M)$ is closed under arbitrary intersection by Proposition 3.17(iii). Nevertheless, if $\xi(M)$ is also closed under finite union, i.e., for any subsemimodules $N_1, \ldots, N_n$ of M, there exists a subsemimodule T of M such that $\bigcup_{i=1}^{n} V'(N_i) = V'(T)$, then $\xi(M)$ satisfies the axioms of closed subsets of any topological space. We call such $\xi(M)$ the ***Zariski topology*** and the R-semimodule M a ***top semimodule***.

Example. (1) Let $R = \mathbb{Z}_0^+$ and $M = \mathbb{Z}_6$. Recall that all subsemimodules of M are $\{\bar{0}\}$, $\{\bar{0},\bar{3}\}$, $\{\bar{0},\bar{2},\bar{4}\}$ and $\mathbb{Z}_6$. It follows that $V'\big(\{\bar{0}\}\big) = \big\{\{\bar{0}\}, \{\bar{0},\bar{3}\}, \{\bar{0},\bar{2},\bar{4}\}\big\}$, $V'\big(\{\bar{0},\bar{3}\}\big) = \big\{\{\bar{0},\bar{3}\}\big\}$, $V'\big(\{\bar{0},\bar{2},\bar{4}\}\big) = \big\{\{\bar{0},\bar{2},\bar{4}\}\big\}$ and $V'\big(\mathbb{Z}_6\big) = \varnothing$. Note that for any subsemimodule N of M,

$$V'\big(\{\bar{0},\bar{3}\}\big) \cup V'\big(\{\bar{0},\bar{2},\bar{4}\}\big) = \big\{\{\bar{0},\bar{3}\}, \{\bar{0},\bar{2},\bar{4}\}\big\} \neq V'(N).$$

Hence $\xi(M)$ is not closed under finite union.

Therefore M is not a top semimodule.

(2) Let $R = \mathbb{Z}_0^+$ and $M = \mathbb{Z}_8$. All subsemimodules of M are $\{\bar{0}\}$, $\{\bar{0},\bar{4}\}$, $\{\bar{0},\bar{2},\bar{4},\bar{6}\}$ and $\mathbb{Z}_8$. Recall that $\{\bar{0}\}$ is not a 2-absorbing subsemimodule by Proposition 2.22. Then $V'\big(\{\bar{0}\}\big) = \big\{\{\bar{0},\bar{4}\}, \{\bar{0},\bar{2},\bar{4},\bar{6}\}\big\}$, $V'\big(\{\bar{0},\bar{4}\}\big) = \big\{\{\bar{0},\bar{4}\}, \{\bar{0},\bar{2},\bar{4},\bar{6}\}\big\}$, $V'\big(\{\bar{0},\bar{2},\bar{4},\bar{6}\}\big) = \big\{\{\bar{0},\bar{2},\bar{4},\bar{6}\}\big\}$ and $V'\big(\mathbb{Z}_8\big) = \varnothing$. Thus $V'\big(\{\bar{0},\bar{4}\}\big) \cup V'\big(\{\bar{0},\bar{2},\bar{4},\bar{6}\}\big) = \big\{\{\bar{0},\bar{4}\}, \{\bar{0},\bar{2},\bar{4},\bar{6}\}\big\} = V'\big(\{\bar{0}\}\big)$. By the same manner, we obtain that there exists a subsemimodule T of M such that $\bigcup_{i=1}^{n} V'(N_i) = V'(T)$ where N_i is a subsemimodule of M. Hence $\xi(M)$ is closed under finite union.

Therefore M is a top semimodule.

The following lamma shows necessary and sufficient conditions of R-semimodule M in order to be a top semimodule.

Lemma 3.18. *Let M be an R-semimodule. Then the following statements are equivalent.*

(i) M is a top semimodule.

(ii) Every 2-absorbing subtractive subsemimodule of M is extraordinary.

(iii) $V'(T) \cup V'(L) = V'(T \cap L)$ for any semi-2-absorbing subtractive subsemimodules T and L of M.

Proof. (i)⇒(ii) Assume that M is a top semimodule. Let N be a 2-absorbing subtractive subsemimodule of M. Let T and L be semi-2-absorbing subtractive subsemimodules of M such that $T \cap L \subseteq N$. Since M is a top semimodule, there exists a subsemimodule U of M such that $V'(T) \cup V'(L) = V'(U)$. Note that $T = \bigcap_{i \in I} N_i$ where N_i is a 2-absorbing subtractive subsemimodule of M for all i because T is semi-2-absorbing. Then $N_i \in V'(T) \subseteq V'(U)$ for each $i \in I$. Thus $U \subseteq N_i$ for all $i \in I$. Hence $U \subseteq T$. Similarly, $U \subseteq L$. This shows that $U \subseteq T \cap L$. By Proposition 3.17(i), we obtain that $V'(T \cap L) \subseteq V'(U)$, $V'(L) \subseteq V'(T \cap L)$ and $V'(T) \subseteq V'(T \cap L)$. Now, we have $V'(T) \cup V'(L) \subseteq V'(T \cap L) \subseteq V'(U) = V'(T) \cup V'(L)$. That is $V'(T) \cup V'(L) = V'(T \cap L)$. Since $T \cap L \subseteq N$, we have $V'(N) \subseteq V'(T \cap L)$. Then $N \in V'(N) \subseteq V'(T \cap L) = V'(T) \cup V'(L)$. Thus $N \in V'(T)$ or $N \in V'(L)$. Hence $T \subseteq N$ or $L \subseteq N$.

Therefore N is extraordinary.

(ii)⇒(iii) Assume that every 2-absorbing subtractive subsemimodule of M is extraordinary. Let T and L be semi-2-absorbing subtractive subsemimodules of M. First, let $A \in V'(T) \cup V'(L)$. Then $T \subseteq A$ or $L \subseteq A$. That is $T \cap L \subseteq A$. Thus $A \in V'(T \cap L)$. Hence $V'(T) \cup V'(L) \subseteq V'(T \cap L)$. Next, let $A \in V'(T \cap L)$. Thus

$T \cap L \subseteq A$. Since A is a 2-absorbing subtractive subsemimodule and by assumption, A is extraordinary. Then $T \subseteq A$ or $L \subseteq A$. Thus $A \in V'(T)$ or $A \in V'(L)$. That is $A \in V'(T) \cup V'(L)$. Hence $V'(T \cap L) \subseteq V'(T) \cup V'(L)$.

Therefore $V'(T) \cup V'(L) = V'(T \cap L)$.

(iii)⇒(i) Assume that $V'(T) \cup V'(L) = V'(T \cap L)$ for any semi-2-absorbing subtractive subsemimodules T and L of M. Let A and B be subsemimodules of M. Then by Proposition 3.17(ii), we get $V'(A) \cup V'(B) = V'\left(\overline{\mathrm{rad}_M(A)}\right) \cup V'\left(\overline{\mathrm{rad}_M(B)}\right)$. Recall that $\overline{\mathrm{rad}_M(A)}$ and $\overline{\mathrm{rad}_M(B)}$ are the intersections of all 2-absorbing subtractive subsemimodules of M containing A and B, respectively. Then $\overline{\mathrm{rad}_M(A)}$ and $\overline{\mathrm{rad}_M(B)}$ are semi-2-absorbing subtractive subsemimodules and $\overline{\mathrm{rad}_M(A)} \cap \overline{\mathrm{rad}_M(B)}$ is a subsemimodule of M. Then by assumption, $V'\left(\overline{\mathrm{rad}_M(A)} \cap \overline{\mathrm{rad}_M(B)}\right) = V'\left(\overline{\mathrm{rad}_M(A)}\right) \cup V'\left(\overline{\mathrm{rad}_M(B)}\right)$. Thus $V'(A) \cup V'(B) = V'\left(\overline{\mathrm{rad}_M(A)} \cap \overline{\mathrm{rad}_M(B)}\right)$. This shows that for any subsemimodules A and B of M there exist a subsemimodule C of M such that $V'(A) \cup V'(B) = V'(C)$. By Mathematical Induction, we conclude that $\xi(M)$ is closed under finite union.

Therefore M is a top semimodule. □

Finally, we study the collection of all 2-absorbing subtractive subsemimodules of multiplication R-semimodule M containing N where N is a subsemimodule of M.

Theorem 3.19. *Let M be an R-semimodule and N, P and L subsemimodules of M. Then the following statements hold.*

(i) $V'(IN) \cup V'(JN) \cup V'(IJM) = V'(IN \cap JN \cap IJM) = V'(IJN)$ *for every ideals I and J of R.*

(ii) $V'(IKM) \cup V'(JKM) \cup V'(IJM) = V'(IKM \cap JKM \cap IJM) = V'(IJKM)$ *for every ideals I, J and K of R.*

(iii) If M is a multiplication R-semimodule, then $V'(NP) \cup V'(LP) \cup V'(NL) = V'(NP \cap LP \cap NL) = V'(NLP)$.

Proof. (i) Let I and J be ideals of R. First, it is clear that $V'(IN) \subseteq V'(IN \cap JN \cap IJM)$, $V'(JN) \subseteq V'(IN \cap JN \cap IJM)$ and $V'(IJM) \subseteq V'(IN \cap JN \cap IJM)$. Thus $V'(IN) \cup V'(JN) \cup V'(IJM) \subseteq V'(IN \cap JN \cap IJM)$. Since $IJN \subseteq IN$, $IJN \subseteq JN$ and $IJN \subseteq IJM$, we get $IJN \subseteq IN \cap JN \cap IJM$. Thus $V'(IN \cap JN \cap IJM) \subseteq V'(IJN)$. Hence $V'(IN) \cup V'(JN) \cup V'(IJM) \subseteq V'(IN \cap JN \cap IJM) \subseteq V'(IJN)$.

Next, let $P \in V'(IJN)$. Thus $IJN \subseteq P$. Since P is a 2-absorbing subtractive subsemimodule, $IN \subseteq P$ or $JN \subseteq P$ or $IJ \subseteq (P : M)$. Then $IN \subseteq P$ or $JN \subseteq P$ or $IJM \subseteq P$. Thus $P \in V'(IN)$ or $P \in V'(JN)$ or $P \in V'(IJM)$. That is $P \in V'(IN) \cup V'(JN) \cup V'(IJM)$. Hence $V'(IJN) \subseteq V'(IN) \cup V'(JN) \cup V'(IJM)$.

Now, we have $V'(IN) \cup V'(JN) \cup V'(IJM) \subseteq V'(IN \cap JN \cap IJM) \subseteq V'(IJN)$ and $V'(IJN) \subseteq V'(IN) \cup V'(JN) \cup V'(IJM)$.

Therefore $V'(IN) \cup V'(JN) \cup V'(IJM) = V'(IN \cap JN \cap IJM) = V'(IJN)$.

(ii) Let I, J and K be ideals of R. Thus KM is a subsemimodule of M. Then by (i), we obtain that $V'(IKM) \cup V'(JKM) \cup V'(IJM) = V'(IKM \cap JKM \cap IJM) = V'(IJKM)$.

(iii) Assume that M is a multiplication R-semimodule. Then there exist ideals I, J and K of R such that $N = IM$, $L = JM$ and $P = KM$. From (ii), we get that $V'(IKM) \cup V'(JKM) \cup V'(IJM) = V'(IKM \cap JKM \cap IJM) = V'(IJKM)$ so that $V'(IMKM) \cup V'(JMKM) \cup V'(IMJM) = V'(IMKM \cap JMKM \cap IMJM) = V'(IMJMKM)$ because M is a multiplication R-semimodule. Thus $V'(NP) \cup V'(LP) \cup V'(NL) = V'(NP \cap LP \cap NL) = V'(NLP)$. □

CHAPTER IV

AG2-ABSORBING SUBSEMIMODULES AND WEAKLY AG2-ABSORBING SUBSEMIMODULES OVER COMMUTATIVE SEMIRINGS

In this section, we extend some characterizations in [5], [10] and [12] to AG2-absorbing and weakly AG2-absorbing subsemimodules over a commutative semiring.

Recall that a proper subsemimodule N of M is said to be an ***almost generalized*** 2***-absorbing subsemimodule*** (or $\boldsymbol{AG}2$***-absorbing subsemimodule***) if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$ and a proper subsemimodule N of M is said to be a ***weakly almost generalized*** 2***-absorbing subsemimodule*** (or ***weakly*** $\boldsymbol{AG}2$***-absorbing subsemimodule***) if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$.

It is clear from the definition that AG2-absorbing subsemimodules are weakly AG2-absorbing subsemimodules. Let us first provide another condition that subsemimodules of an R-semimodule are weakly AG2-absorbing subsemimodules.

Proposition 4.1. *Let M be an R-semimodule and N a proper subsemimodule of M. If $\sqrt{(N : m)} = \sqrt{(N : M)} \cup (\{0\} : m)$ for all $m \in M \setminus N$, then N is a weakly $AG2$-absorbing subsemimodule of M.*

Proof. Assume that $\sqrt{(N : m)} = \sqrt{(N : M)} \cup (\{0\} : m)$ for all $m \in M \setminus N$. Let $a, b \in R$ and $m \in M$ be such that $0 \neq abm \in N$ but $am \notin N$ and $bm \notin N$. Then $m \notin N$. Thus $\sqrt{(N : m)} = \sqrt{(N : M)} \cup (\{0\} : m)$. The fact that $abm \in N$ implies $ab \in (N : m) \subseteq \sqrt{(N : m)}$ and $ab \notin (\{0\} : m)$ because $0 \neq abm \in N$. This forces $ab \in \sqrt{(N : M)}$.

Therefore N is a weakly AG2-absorbing subsemimodule. $\square$

However, weakly AG2-absorbing subsemimodules need not be AG2-absorbing subsemimodules in general. Hence some conditions are needed to make weakly AG2-absorbing subsemimodules and AG2-absorbing subsemimodules be identical.

Theorem 4.2. *Let M be an R-semimodule and N a weakly AG2-absorbing subsemimodule of M. If N is a subtractive subsemimodule and $(N : M)^2N \neq \{0\}$, then N is an AG2-absorbing subsemimodule.*

Proof. Assume that N is a subtractive subsemimodule and $(N : M)^2N \neq \{0\}$. Proposition 2.13 provides that $(N : M)$ is a subtractive ideal of R. Let $a, b \in R$ and $m \in M$ be such that $abm \in N$. We claim that $am \in N$ or $bm \in N$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$. If $0 \neq abm \in N$, then we are through because N is a weakly AG2-absorbing subsemimodule of M. Then assume that $abm = 0$.
Case 1: $abN \neq \{0\}$. Then there is $n_0 \in N$ such that $abn_0 \neq 0$. Now $0 \neq abn_0 = 0 + abn_0 = abm + abn_0 = ab(m + n_0) \in N$. Since N is a weakly AG2-absorbing subsemimodule, $a(m + n_0) \in N$ or $b(m + n_0) \in N$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$. If $a(m + n_0) \in N$ or $b(m + n_0) \in N$, then applying the fact that N is subtractive leads to $am \in N$ or $bm \in N$. Thus $am \in N$ or $bm \in N$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$.
Case 2: $abN = \{0\}$.

Subcase 2.1: $a(N : M)M \neq \{0\}$ or $b(N : M)M \neq \{0\}$. Without loss of generality, assume that $a(N : M)M \neq \{0\}$. Then there exists $r \in (N : M)$ such that $arm \neq 0$. Thus $0 \neq arm = abm + arm = a(b+r)m \in N$. Since N is a weakly AG2-absorbing, $am \in N$ or $(b+r)m \in N$ or $[a(b+r)]^k \in (N : M)$ for some $k \in \mathbb{N}$. If $(b + r)m \in N$ or $[a(b + r)]^k \in (N : M)$ for some $k \in \mathbb{N}$, then applying the fact that N and $(N : M)$ are subtractive leads to $bm \in N$ or $(ab)^k \in (N : M)$. Thus $am \in N$ or $bm \in N$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$.

Subcase 2.2: $a(N : M)M = \{0\}$ and $b(N : M)M = \{0\}$. Since $(N : M)^2N \neq \{0\}$, there exist $a_0, b_0 \in (N : M)$ and $x_0 \in N$ with $0 \neq a_0b_0x_0 \in N$. Then $a_0b_0m \in N$ and $abx_0 = 0$. Moreover, $ab_0m = 0$, $a_0bm = 0$, $ab_0x_0 = 0$ and

$a_0bx_0 = 0$.

Subcase 2.2.1: $a_0b_0m \neq 0$. Then $0 \neq a_0b_0m = abm + ab_0m + a_0bm + a_0b_0m = (a+a_0)(b+b_0)m$. In addition, $(a+a_0)(b+b_0)m \in N$ because $a_0b_0m \in N$. Since N is a weakly AG2-absorbing and $0 \neq (a + a_0)(b + b_0)m \in N$, we obtain that $(a + a_0)m \in N$ or $(b + b_0)m \in N$ or $[(a + a_0)(b + b_0)]^k \in (N : M)$ for some $k \in \mathbb{N}$. Thus $am + a_0m \in N$ or $bm + b_0m \in N$ or $a^kb^k + r \in (N : M)$ for some $r \in (N : M)$. To see this, we consider $[(a + a_0)(b + b_0)]^k$ as follows:

$$\begin{aligned}
&[(a + a_0)(b + b_0)]^k \\
&= a^kb^k + a^k\sum_{i=1}^{k}\binom{k}{i} b^{k-i}b_0^i + \binom{k}{1} a^{k-1}a_0\sum_{i=0}^{k}\binom{k}{i} b^{k-i}b_0^i \\
&\quad + \binom{k}{2} a^{k-2}a_0^2\sum_{i=0}^{k}\binom{k}{i} b^{k-i}b_0^i + \cdots + \binom{k}{j} a^{k-j}a_0^j\sum_{i=0}^{k}\binom{k}{i} b^{k-i}b_0^i + \cdots \\
&\quad + a_0^k\sum_{i=0}^{k}\binom{k}{i} b^{k-i}b_0^i \quad \text{where } j \in \{0, 1, 2, \ldots, k\}.
\end{aligned}$$

Let $r = a^k\sum_{i=1}^{k}\binom{k}{i} b^{k-i}b_0^i + \binom{k}{1} a^{k-1}a_0\sum_{i=0}^{k}\binom{k}{i} b^{k-i}b_0^i + \binom{k}{2} a^{k-2}a_0^2\sum_{i=0}^{k}\binom{k}{i} b^{k-i}b_0^i + \cdots + \binom{k}{j} a^{k-j}a_0^j\sum_{i=0}^{k}\binom{k}{i} b^{k-i}b_0^i + \cdots + a_0^k\sum_{i=0}^{k}\binom{k}{i} b^{k-i}b_0^i$. Then $r \in (N : M)$ because $a_0, b_0 \in (N : M)$. Moreover, since $a_0, b_0 \in (N : M)$, it follows that $a_0m, b_0m \in N$. Being subtractive of N and $(N : M)$ implies that $am \in N$ or $bm \in N$ or $(ab)^k = a^kb^k \in (N : M)$.

Subcase 2.2.2: $a_0b_0m = 0$. Then $0 \neq a_0b_0x_0 = abm + abx_0 + ab_0m + ab_0x_0 + a_0bm + a_0bx_0 + a_0b_0m + a_0b_0x_0 = (a + a_0)(b + b_0)(m + x_0)$. We obtain that $0 \neq (a + a_0)(b + b_0)(m + x_0) \in N$ because $0 \neq a_0b_0x_0 \in N$. Since N is weakly AG2-absorbing, $(a + a_0)(m + x_0) \in N$ or $(b + b_0)(m + x_0) \in N$ or $[(a + a_0)(b + b_0)]^k \in (N : M)$ for some $k \in \mathbb{N}$.

Subcase 2.2.2.1: $(a+a_0)(m+x_0) \in N$. Then $am+ax_0+a_0m+a_0x_0 = (a + a_0)(m + x_0) \in N$. Since N is a subtractive subsemimodule, $am \in N$.

Subcase 2.2.2.2: $(b+b_0)(m+x_0) \in N$. Similarly to Subcase 2.2.2.1, we obtain that $bm \in N$.

Subcase 2.2.2.3: $[(a+a_0)(b+b_0)]^k \in (N:M)$ for some $k \in \mathbb{N}$. Then $[(a+a_0)(b+b_0)]^k = a^k b^k + r$ for some $r \in (N:M)$. Since $(N:M)$ is subtractive, $(ab)^k = a^k b^k \in (N:M)$.

Therefore N is an AG2-absorbing subsemimodule. □

Corollary 4.3. *Let M be an R-semimodule and N a proper subsemimodule of M. If $\sqrt{(N:m)} = \sqrt{(N:M)} \cup (\{0\}:m)$ for all $m \in M \setminus N$ and N is a subtractive subsemimodule with $(N:M)^2 N \neq \{0\}$, then N is an AG2-absorbing subsemimodule.*

Proof. Assume that $\sqrt{(N:m)} = \sqrt{(N:M)} \cup (\{0\}:m)$ for all $m \in M \setminus N$ and N is a subtractive subsemimodule with $(N:M)^2 N \neq \{0\}$. We obtain from Proposition 4.1 that N is a weakly AG2-absorbing subsemimodule. Then N is an AG2-absorbing subsemimodule by Theorem 4.2. □

The next proposition provides the condition for $\sqrt{(N:M)}$ to be a 2-absorbing ideal of R where N is a subsemimodule of an R-semimodule M. However, the R-semimodule M has to be cyclic.

Proposition 4.4. *Let M be a cyclic R-semimodule and N a subsemimodule of M. If N is an AG2-absorbing subsemimodule of M, then $\sqrt{(N:M)}$ is a 2-absorbing ideal of R containing $\operatorname{ann}(M)$.*

Proof. First of all, let $M = Rm$ for some $m \in M$ and assume that N is an AG2-absorbing subsemimodule of M. Recall that $(N:M) = (N:m)$. To show that $\sqrt{(N:M)}$ is a 2-absorbing ideal, let $a, b, c \in R$ be such that $abc \in \sqrt{(N:M)}$ but $ab \notin \sqrt{(N:M)}$ and $ac \notin \sqrt{(N:M)}$. Then there exists $k \in \mathbb{N}$ such that $a^k b^k c^k \in (N:M)$, i.e., $a^k b^k c^k M \subseteq N$. Since $ab \notin \sqrt{(N:M)}$ and $ac \notin \sqrt{(N:M)}$, there are $r_1, r_2 \in R$ such that $(ab)^k r_1 m, (ac)^k r_2 m \notin N$ so that $a^k b^k r_1 m, a^k c^k r_2 m \notin N$. Thus $a^k b^k m \notin N$ and $a^k c^k m \notin N$. Since N is an AG2-absorbing subsemimodule and $b^k c^k (a^k m) \in N$, it follows that $b^k a^k m \in N$ or $c^k a^k m \in N$ or $(b^k c^k)^l \in (N:M)$ for some $l \in \mathbb{N}$. As a result, $bc \in \sqrt{(N:M)}$ because $b^k a^k m \notin N$ and $c^k a^k m \notin N$.

Next, we show that $\mathrm{ann}(M) \subseteq \sqrt{(N:M)}$. We know that $\mathrm{ann}(M) = (\{0\}:M)$. Then $\mathrm{ann}(M) = (\{0\}:M) \subseteq (N:M) \subseteq \sqrt{(N:M)}$ as desired.

Therefore $\sqrt{(N:M)}$ is a 2-absorbing ideal of R containing $\mathrm{ann}(M)$. □

The followings provide some relationships between being (weakly) AG2-absorbing subsemimodules of N and being (weakly) AG2-absorbing ideals of $(N:M)$ where N is a subsemimodule of a cyclic R-semimodule M.

Proposition 4.5. *Let M be a cyclic R-semimodule and N a subsemimodule of M. Then the followings hold.*

(i) N is an AG2-absorbing subsemimodule of M if and only if $(N:M)$ is an AG2-absorbing ideal of R.

(ii) If, in addition, M is faithful, then N is a weakly AG2-absorbing subsemimodule of M if and only if $(N:M)$ is a weakly AG2-absorbing ideal of R.

Proof. Let $M = Rm$ for some $m \in M$. Then $(N:M) = (N:m)$.

(i) First, assume that N is an AG2-absorbing subsemimodule of M. Let $a, b, c \in R$ be such that $abc \in (N:M)$ but $ab \notin (N:M)$ and $ac \notin (N:M)$. Then there exist $r, s \in R$ such that $ab(rm) \notin N$ and $ac(sm) \notin N$. Thus $abm \notin N$ and $acm \notin N$. Note that $bc(am) \in N$. Thus there exists $k \in \mathbb{N}$ such that $(bc)^k \in (N:M)$ because N is AG2-absorbing.

Therefore $(N:M)$ is an AG2-absorbing ideal of R.

Conversely, assume that $(N:M)$ is an AG2-absorbing ideal of R. Let $a, b \in R$ and $x \in M$ be such that $abx \in N$. Then there exists $c \in R$ such that $x = cm$, so $abcm \in N$, i.e., $abc \in (N:m) = (N:M)$. Thus $ac \in (N:m)$ or $bc \in (N:m)$ or $(ab)^k \in (N:M)$ for some $k \in \mathbb{N}$, i.e., $ax = acm \in N$ or $bx = bcm \in N$ or $(ab)^k \in (N:M)$.

Therefore N is an AG2-absorbing subsemimodule of M.

(ii) Assume further that M is faithful.

First, let N be a weakly AG2-absorbing subsemimodule of M. Let $a, b, c \in R$

be such that $0 \neq abc \in (N : M)$ but $ab \notin (N : M)$ and $ac \notin (N : M)$. Then there exist $r, s \in R$ such that $ab(rm) \notin N$ and $ac(sm) \notin N$. Thus $abm \notin N$ and $acm \notin N$. Suppose that $abcm = 0$. So, $\{0\} = abcRm = abcM$. Since M is faithful, $abc = 0$ which is a contradiction. Thus $abcm \neq 0$. Now, $0 \neq abcm \in N$. Since N is a weakly AG2-absorbing subsemimodule, $b(am) \notin N$ and $c(am) \notin N$, we obtain that $(bc)^k \in (N : M)$ for some $k \in \mathbb{N}$.

Therefore $(N : M)$ is a weakly AG2-absorbing ideal of R.

Next, let $(N : M)$ be a weakly AG2-absorbing ideal of R. Let $a, b \in R$ and $x \in M$ be such that $0 \neq abx \in N$. Then there exists $c \in R$ such that $x = cm$, so $0 \neq abcm \in N$. Thus $0 \neq abc \in (N : m) = (N : M)$ so that $ac \in (N : m)$ or $bc \in (N : m)$ or $(ab)^k \in (N : M)$ for some $k \in \mathbb{N}$. Hence $ax = acm \in N$ or $bx = bcm \in N$ or $(ab)^k \in (N : M)$.

Therefore N is a weakly AG2-absorbing subsemimodule of M. □

This chapter is ended by providing relationship between (weakly) AG2-absorbing subsemimodules and (weakly) AG2-absorbing Q-subsemimodules. This proof is quite similar to the proof of Propositon 3.2 and Propositon 3.3 in Chapter III.

Theorem 4.6. *Let M be an R-semimodule, N a Q-subsemimodule of M and P a subtractive subsemimodule of M with $N \subseteq P$. Then P is an AG2-absorbing subsemimodule of M if and only if $P/N_{(Q\cap P)}$ is an AG2-absorbing subsemimodule of $M/N_{(Q)}$.*

Proof. First, assume that P is an AG2-absorbing subsemimodule of M. Then $P/N_{(Q\cap P)}$ is a proper subsemimodule of $M/N_{(Q)}$. Let $a, b \in R$ and $q_1 + N \in M/N_{(Q)}$, where $q_1 \in Q$, be such that $ab \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Then there exists unique $q_2 \in Q \cap P$ such that $ab \odot (q_1 + N) = q_2 + N$ where $abq_1 + N \subseteq q_2 + N$. Since $q_2 + N \subseteq P$, it follows that $abq_1 + N \subseteq P$. Since $N \subseteq P$ and P is subtractive, $abq_1 \in P$. Since P is AG2-absorbing, $aq_1 \in P$ or $bq_1 \in P$ or $(ab)^k M \subseteq P$ for some $k \in \mathbb{N}$. We claim that $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $(ab)^k \in (P/N_{(Q\cap P)} : M/N_{(Q)})$.

Case 1: $aq_1 \in P$. Then $aq_1 \in Q \cap P$, so $aq_1 + N \in P/N_{(Q\cap P)}$. Moreover,

$a \odot (q_1 + N) = q' + N$ where $q' \in Q$ is unique such that $aq_1 + N \subseteq q' + N$. Then $q' = aq_1 \in Q \cap P$. Thus $a \odot (q_1 + N) \in P/N_{(Q \cap P)}$.

Case 2: $bq_1 \in P$**.** We can conclude similarly to Case 1 that $b \odot (q_1 + N) \in P/N_{(Q \cap P)}$.

Case 3: $(ab)^k M \subseteq P$**.** Let $q + N \in M/N_{(Q)}$ and $(ab)^k \odot (q + N) = q_3 + N$ where $q_3 \in Q$ is unique such that $(ab)^k q + N \subseteq q_3 + N$. Then $(ab)^k q + N = q_3 + N$ since $(ab)^k q \in Q$. Thus $q_3 + N = (ab)^k q + N \subseteq P$ since $(ab)^k M \subseteq P$ and $N \subseteq P$ so that $q_3 \in P$ because P is subtractive. As a result, $q_3 \in Q \cap P$. Then $(ab)^k \odot (q + N) = q_3 + N \in P/N_{(Q \cap P)}$. Thus $(ab)^k \odot M/N_{(Q)} \subseteq P/N_{(Q \cap P)}$. Hence $(ab)^k \in (P/N_{(Q \cap P)} : M/N_{(Q)})$.

Therefore $P/N_{(Q \cap P)}$ is an AG2-absorbing subsemimodule of $M/N_{(Q)}$.

Conversely, assume that $P/N_{(Q \cap P)}$ is an AG2-absorbing subsemimodule of $M/N_{(Q)}$. Then P is a proper subsemimodule of M. Let $a, b \in R$ and $m \in M$ be such that $abm \in P$. Then by Proposition 2.25, there is unique $q_1 \in Q$ such that $m \in q_1 + N$ and $abm \in ab \odot (q_1 + N)$. Let $ab \odot (q_1 + N) = q_2 + N$ where q_2 is a unique element of Q such that $abq_1 + N \subseteq q_2 + N$. Now, $abm \in P$ and $abm \in q_2 + N$. So there is $n \in N$ such that $q_2 + n = abm \in P$. Since P is subtractive and $n \in N \subseteq P$, we obtain $q_2 \in P$. Then $q_2 \in Q \cap P$. Thus $ab \odot (q_1 + N) = q_2 + N \in P/N_{(Q \cap P)}$. Since $P/N_{(Q \cap P)}$ is an AG2-absorbing subsemimodule, $a \odot (q_1 + N) \in P/N_{(Q \cap P)}$ or $b \odot (q_1 + N) \in P/N_{(Q \cap P)}$ or $(ab)^k \odot M/N_{(Q)} \subseteq P/N_{(Q \cap P)}$ for some $k \in \mathbb{N}$.

Case 1: $a \odot (q_1 + N) \in P/N_{(Q \cap P)}$**.** Then $a \odot (q_1 + N) = q' + N$ where q' is a unique element of $Q \cap P$ such that $aq_1 + N \subseteq q' + N$. Since $a \odot (q_1 + N) \subseteq P$ and $a \odot (q_1 + N) = q' + N$, we get $aq_1 + N \subseteq P$. Thus $aq_1 \in P$. Then $aq_1 \in Q \cap P$. So $q' = aq_1$. It follows from $m \in q_1 + N$ that $am \in a(q_1 + N) \subseteq aq_1 + N = q' + N = a \odot (q_1 + N) \subseteq P$. Thus $am \in P$.

Case 2: $b \odot (q_1 + N) \in P/N_{(Q \cap P)}$**.** Again, this is similar to Case 1, we can conclude that $bm \in P$.

Case 3: $(ab)^k M/N_{(Q)} \subseteq P/N_{(Q \cap P)}$**.** Let $x \in M$. By Proposition 2.25, there is unique $q_3 \in Q$ such that $x \in q_3 + N$ and $(ab)^k x \in (ab)^k \odot (q_3 + N) = q_4 + N$ where q_4 is a unique element of Q such that $(ab)^k q_3 + N \subseteq q_4 + N$. Now,

$q_4 + N = (ab)^k \odot (q_3 + N) \in P/N_{(Q\cap P)}$. Then $(ab)^k x \in q_4 + N \subseteq P$. Thus $(ab)^k M \subseteq P$.

Therefore P is an AG2-absorbing subsemimodule of M. □

Theorem 4.7. *Let M be an R-semimodule, N a Q-subsemimodule of M and P a subtractive subsemimodule of M with $N \subseteq P$.*

(i) If P is a weakly AG2-absorbing subsemimodule of M, then $P/N_{(Q\cap P)}$ is a weakly AG2-absorbing subsemimodule of $M/N_{(Q)}$.

(ii) If N and $P/N_{(Q\cap P)}$ are weakly AG2-absorbing subsemimodules of M and $M/N_{(Q)}$, respectively, then P is a weakly AG2-absorbing subsemimodule of M.

Proof. (i) Assume that P is a weakly AG2-absorbing subsemimodule of M. Then $P/N_{(Q\cap P)}$ is a proper subsemimodule of $M/N_{(Q)}$. Let $a, b \in R$ and $q_1 + N \in M/N_{(Q)}$, where $q_1 \in Q$, be such that $0_M + N \neq ab \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Then there exists unique $q_2 \in Q \cap P$ such that $ab \odot (q_1 + N) = q_2 + N$ where $abq_1 + N \subseteq q_2 + N$. Since $q_2 + N \subseteq P$, it follows that $abq_1 + N \subseteq P$ and then $abq_1 \in P$.

Case 1: $abq_1 = 0$. Since $abq_1 \in (0_M + N) \cap (q_2 + N)$, we obtain that $0_M = q_2$. Thus $0_M + N = q_2 + N$ contradicts the fact that $q_2 + N = ab \odot (q_1 + N) \neq 0_M + N$.

Case 2: $abq_1 \neq 0$. Since P is a weakly AG2-absorbing subsemimodule of M, it can be concluded that $aq_1 \in P$ or $bq_1 \in P$ or $(ab)^k M \subseteq P$ for some $k \in \mathbb{N}$. We claim that $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $(ab)^k \in (P/N_{(Q\cap P)} : M/N_{(Q)})$.

Subcase 2.1: $aq_1 \in P$**.** Then $aq_1 \in Q \cap P$, so $aq_1 + N \in P/N_{(Q\cap P)}$. Moreover, $a \odot (q_1 + N) = q' + N$ where $q' \in Q$ is unique such that $aq_1 + N \subseteq q' + N$. Then $q' = aq_1 \in Q \cap P$. Thus $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$.

Subcase 2.2: $bq_1 \in P$**.** We can conclude similarly to Subcase 2.1 that $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$.

Subcase 2.3: $(ab)^k M \subseteq P$**.** Let $q + N \in M/N_{(Q)}$ and $(ab)^k \odot (q + N) = q_3 + N$ where $q_3 \in Q$ is unique such that $(ab)^k q + N \subseteq q_3 + N$. Then $(ab)^k q + N = q_3 + N$ since $(ab)^k q \in Q$. Then $q_3 + N = (ab)^k q + N \subseteq P$ since $(ab)^k M \subseteq P$ and

$N \subseteq P$ so that $q_3 \in P$ because P is subtractive. Thus $q_3 \in Q \cap P$. Then $(ab)^k \odot (q + N) = q_3 + N \in P/N_{(Q\cap P)}$. Thus $(ab)^k \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$. Hence $(ab)^k \in (P/N_{(Q\cap P)} : M/N_{(Q)})$.

Therefore $P/N_{(Q\cap P)}$ is a weakly AG2-absorbing subsemimodule of $M/N_{(Q)}$.

(ii) Assume that N and $P/N_{(Q\cap P)}$ are weakly AG2-absorbing subsemimodules of M and $M/N_{(Q)}$, respectively. Then P is a proper subsemimodule of M. Let $a, b \in R$ and $m \in M$ be such that $0 \neq abm \in P$.

Case 1: $0 \neq abm \in N$. Then $am \in N \subseteq P$ or $bm \in N \subseteq P$ or $(ab)^k \in (N : M) \subseteq (P : M)$ for some $k \in \mathbb{N}$.

Case 2: $0 \neq abm \in P \setminus N$. Then by Proposition 2.25, there is unique $q_1 \in Q$ such that $m \in q_1 + N$ and $abm \in ab \odot (q_1 + N)$. Let $ab \odot (q_1 + N) = q_2 + N$ where q_2 is a unique element of Q such that $abq_1 + N \subseteq q_2 + N$. Now, $abm \in P$ and $abm \in q_2 + N$. So there is $n \in N$ such that $q_2 + n = abm \in P$. Since P is subtractive and $n \in N \subseteq P$, we obtain $q_2 \in P$. Then $q_2 \in Q \cap P$. Assume that $0_M + N = ab \odot (q_1 + N)$. Since $q_2 + N = ab \odot (q_1 + N) = 0_M + N$ and $abm \in q_2 + N$, it follows that $abm \in 0_M + N = N$ contradicts the fact that $abm \in P \setminus N$. Thus $0_M + N \neq ab \odot (q_1 + N) = q_2 + N \in P/N_{(Q\cap P)}$. Since $P/N_{(Q\cap P)}$ is a weakly AG2-absorbing subsemimodule, $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $(ab)^k \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$ for some $k \in \mathbb{N}$.

Subcase 2.1: $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Then $a \odot (q_1 + N) = q' + N$ where q' is a unique element of $Q \cap P$ such that $aq_1 + N \subseteq q' + N$. Since $a \odot (q_1 + N) \subseteq P$ and $a \odot (q_1 + N) = q' + N$, we get that $aq_1 + N \subseteq P$. Thus $aq_1 \in P$ because P is subtractive and $N \subseteq P$. Then $aq_1 \in Q \cap P$. So $q' = aq_1$. Since $m \in q_1 + N$, it follows that $am \in a(q_1 + N) \subseteq aq_1 + N = q' + N = a \odot (q_1 + N) \subseteq P$. Thus $am \in P$.

Subcase 2.2: $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Again, this is similar to Subcase 2.1, we can conclude that $bm \in P$.

Subcase 2.3: $(ab)^k M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$. Let $x \in M$. By Proposition 2.25, there is unique $q_3 \in Q$ such that $x \in q_3 + N$ and $(ab)^k x \in (ab)^k \odot (q_3 + N) = q_4 + N$

where q_4 is a unique element of Q such that $(ab)^k q_3 + N \subseteq q_4 + N$. Now, $q_4 + N = (ab)^k \odot (q_3 + N) \in P/N_{(Q \cap P)}$. Then $(ab)^k x \in q_4 + N \subseteq P$. Thus $(ab)^k M \subseteq P$.

Therefore P is a weakly AG2-absorbing subsemimodule of M. $\square$

REFERENCES

[1] Anderson, D.D. and Smith, E.: Weakly prime ideals, *Houston J. Math.* **29**, 831–840(2003).

[2] Atani, R.E. and Atani, S.E.: On subsemimodules of semimodules, *Bul. Acad. Siinte Repub. Mold. Mat.* **2**, 20–30(2010).

[3] Atani, S.E., Atani, R.E. and Tekir, U.: A Zariski topology for semimodules, *Eur. J. Pure Appl. Math.* **4**, 251–265(2011).

[4] Atani, S.E. and Farzalipour, F.: On weakly prime submodules, *Tamkang J. Math.* **38**, 247–252(2007).

[5] Atani, S.E. and Kohan, M.S.: A note on finitely generated multiplication semimodules over commutative semirings, *Int. J. Algebra* **4**, 389–396(2010).

[6] Badawi, A.: On 2-absorbing ideals of commutative rings, *Bull. Austral. Math. Soc.* **75**, 417–429(2007).

[7] Badawi, A. and Darani, A.Y.: On weakly 2-absorbing ideals of commutative rings, *Houston J. Math.*, To appear.

[8] Chaudhari, J.N.: 2-absorbing ideals in semirings, *Int. J. Algebra* **6**, 265–270(2012).

[9] Chaudhari, J.N. and Bonde, B.R.: On partitioning and subtractive subsemimodules of semimodules over semirings, *Kyungpook Math. J.* **50**, 329–336(2010).

[10] Chaudhari, J.N. and Bonde, B.R.: Weakly prime subsemimodules of semimodules over semirings, *Int. J. Algebra* **5**, 167–174(2011).

[11] Darani, A.Y.: On 2-absorbing and weakly 2-absorbing ideals of commutative semirings, *Kyungpook Math. J.* **52**, 91–97(2012).

[12] Darani, A.Y. and Soheilnia, F.: 2-absorbing and weakly 2-absorbing submodules, *Thai. J. Math.* **9**, 577–584(2011).

[13] Golan, J.S.: *Semirings and their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.

[14] Yesilot, G., Oral, K.H. and Tekir, U.: On prime subsemimodules of semimodules, *Int. J. Algebra* **4**, 53–60(2010).

VITA

Name Miss Issaraporn Thongsomnuk

Date of Birth 12 November 1988

Place of Birth Phetchaburi, Thailand

Education B.Sc. (Mathematics)(First Class Honours),
Kasetsart University, 2010

Scholarship Development and Promotion of Science and Technology
Talents Project (DPST)

Conference **Speaker**

- *Symmetric Functions of the Roots of a Polynomial Equations* at the 6$^{\text{th}}$ Science and Technology Conference for Youths, 18-19 March 2011 at BITEC
- *Almost Generalized 2-Absorbing and Weakly Almost Generalized 2-Absorbing Subsemimodules over Commutative Semirings* at the 18$^{\text{th}}$ Annual Meeting in Mathematics, 14-16 March 2013 at Aonang Ayodhaya Beach Resort, Krabi
- *Almost Generalized 2-Absorbing and Weakly Almost Generalized 2-Absorbing Subsemimodules over Commutative Semirings* at the 8$^{\text{th}}$ Science and Technology Conference for Youths, 22-23 March 2013 at BITEC