Chapter 1
Introduction

In this chapter, we give some background in Bose-Einstein condensation (BEC) and explain the scope of the work in this thesis.

1.1 Basic Physics of Bose-Einstein Condensation

The first realization of the Bose-Einstein condensation in trapped dilute gas of alkali atoms in the year 1995 [1,2,3] leads to a new understanding of the superfluid systems. This system is very interesting because of the diluteness and the small interactions between atoms so that we can use perturbative theories to study it. In this chapter we give a brief introduction to the subject and the scope of this thesis is stated.

1.1.1 Einstein's Prediction for the Ideal Bose Gas

Einstein considered N non-interacting bosonic and non-relativistic particles in a cubic box of volume L^3 with periodic boundary conditions. In the thermodynamic limit, defined as

$$N, L \to \infty, \tag{1.1}$$

where

$$\frac{N}{L^3} = \rho = \text{finite}, \tag{1.2}$$

a phase transition occurs at a temperature T_c defined by

$$\rho\lambda^3 (T_c) = \zeta (3/2) = 2.612... \tag{1.3}$$

where we have defined the thermal de Broglie wavelength of the gas as a function of the temperature T,

$$\lambda (T) = \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{\frac{1}{2}} \tag{1.4}$$

and $\zeta (a) = \sum_{k=1}^{\infty} 1/k^a$ is the Riemann Zeta function.

The order parameter of this phase transition is the fraction N_0/N of the particles in the ground state of the box, that is, the plane wave with momentum $\vec{p} = \vec{0}$. For temperatures lower than T_c this fraction N_0/N remains finite in the thermodynamic limit, whereas it tends to zero when $T > T_c$:

For $T > T_c$,

$$N_0/N \to 0; \tag{1.5}$$

For $T < T_c$,

$$N_0/N \sim \left(1 - \left(\frac{T}{T_c} \right)^{3/2} \right). \tag{1.6}$$

For $T < T_c$ the system has formed a Bose-Einstein condensate in the $\vec{p} = \vec{0}$ state. The number N_0 of particles in the condensate is on the order of N, which is macroscopic. As we will see, the macroscopic population of a single quantum state is the key feature of a Bose-Einstein condensate, and gives rise to interesting properties such as coherence. For more details, see for example [4]

1.1.2 Experimental Realization of BEC in Trapped Gas

The major problem encountered experimentally to verify Einstein's predictions is that at densities and temperatures required by Eq. (1.3) at thermodynamic equilibrium almost all materials are in the solid state. An exception is ^{4}He which is a fluid at $T = 0$. However ^{4}He is a strongly interacting system. In ^{4}He in sharp contrast with the prediction for the ideal gas, Eq. (1.6), $N_0/N < 10\%$ even at zero temperature. The solution which successfully led to Bose-Einstein condensation in atomic gases is to bring the system to extremely low densities (much lower than in a normal gas) and to cool it rapidly enough so that it has no time to recombine and solidify. The price to pay for an ultralow density is the necessity to cool at extremely low kinetic temperatures. Typically one has in the experiments with condensates:

$$\rho < 10^{15} \ \text{atoms/cm}^3 \tag{1.7}$$

and

$$T < 1 \ \mu\text{K}. \tag{1.8}$$

The critical temperatures range from 20 nK to the μK range. Bose-Einstein condensation was achieved for the first time in atomic gases in 1995. The group of Eric Cornell and Carl Wieman at JILA was successful with ^{87}Rb atoms [1]. They were closely followed by the group of Wolfgang Ketterle at MIT with ^{23}Na atoms [2] and the group of Randy Hulet at Rice University with ^{7}Li atoms [3]. Nowadays people have obtained many condensates mainly with rubidium or sodium atoms. No other alkali atoms than the ones of the year 1995 have been condensed. Atomic hydrogen has been condensed in 1998 at MIT by the group of Dan Kleppner [5]; the experiments on hydrogen were actually the first ones to start and played a fundamental pioneering role in developing many of the experimental techniques having led to success with the alkali atoms , such as magnetic trapping and evaporative cooling of atoms.

1.1.3 The Reasons for Studying the BEC

An important theoretical framework for Bose-Einstein condensation in interacting systems was developed in the 50's by Beliaev, Bogoliubov, Gross and Pitaevskii in the context of superfluid helium. This theory however is supposed to work better if applied to Bose condensed gases where the interactions are much weaker. The interactions in ultracold atomic gases can be described by a single parameter a, the so-called scattering length, as interactions take place between atoms with very low relative kinetic energy. The gaseous condensates are dilute systems as the mean interparticle separation is much larger than the scattering length a,

$$\rho \left| a \right|^3 \ll 1. \tag{1.9}$$

This provides a small parameter for the theory and, as we shall see, the simple mean field approaches can be used with success to describe most of the properties of the atomic condensates.

Atomic gases offer some new interesting features with respect to superfluid helium 4:

- *Spatial inhomogeneity*: This feature can be used as a tool to detect the presence of a Bose-Einstein condensate inside the trap: in an inhomogeneous gas Bose-Einstein condensation occurs not only in momentum space but also in position space

- *Finite size effects*: The number of atoms in condensates of alkali gases is usually $N_0 < 10^7$. The hydrogen condensate obtained at MIT by Kleppner is larger with $N_0 \sim 10^9$. Interesting finite size effects, that is the effects which disappear in the thermodynamic limit, such as Bose-Einstein condensates with effective attractive interactions ($a < 0$), can be studied in relatively small condensates.

1.2 Scope of the Work in this Thesis

The path integral formulation for statistical mechanics created by Feynman [5] provides the method for calculating the thermodynamic properties of many-particles systems. In this formulation one has to make the sum over all permutations incorporate the Bose-Einstein or Fermi-Dirac statistics into the calculation. However this method has not been used extensively. Recently, Brosens et al. [7,8] used the idea in [6] to formulate a calculation scheme for thermodynamical systems of many bosons or fermions and applied it to the systems of trapped bosons interacting with a harmonic potential. Technically, this model, giving rise to repetitive Gaussian integrals, allows one to derive an analytical expression for the generating function of the partition function. For an ideal gas of noninteracting particles in a parabolic well, this generating function coincides with the grand-canonical partition function. With interactions, the calculation of this generating function circumvents the constraints on the summation over the cycles of the permutation group. Moreover, it allows one to calculate the canonical partition function recursively for the system with harmonic two-body interactions. The permutation symmetry leads to summations over the cycles that are performed using the generating function technique.

From the point of view of path integral techniques, it is interesting to utilize the many-body path integration to the system of confined bosons in a harmonic trap because the simplicity of the system can be a testing ground of the method. The work presented in this thesis is as follows.

1.2.1 Calculation of the Ground State Energy

In Chapter 3, we devise a simple variational method to estimate the ground state energy of the condensate. By using the generating functional technique we obtain the density matrix of the system. The ground state wave function also comes out naturally when we consider the density matrix in the zero temperature limit. The advantage of this method is the analytical result which can be found for various forms of interaction between particles. The results are compared to those obtained by the mean-field method. However, this method has some limitations and it will be discussed in Chapter 3.

1.2.2 Collective Excitation in BEC

Right after the realizations of the Bose-Einstein condensation in a trapped gas, the dynamical aspects of the system were studied [9,10]. This is the so-called collective excitations or collective modes of the condensate. In the experiment, it was found, after perturbing the system by applying time-depending force or by changing the trap strength, that the condensate oscillated with time.

In Chapter 4, we use the technique developed by Brosens et al. [7,8] to find the time evolution of the system by studying the density. The simplest case we calculate is the non-interacting many-particle system. We find that the oscillation frequency of the condensate is in agreement with the experimental or other theoretical results. The study of these first two topics leads to the last topic in this thesis.

1.2.3 Vortex Precession and the Collective Excitation

From the experimental study of the condensate with vortex [11] the precession of the vortex was found. This is due to the Magnus force on the vortex. This phenomenon was simulated successfully within the model of Brosens et al. [12]. Hence, it is a good idea to study the condensate with vortex being perturbed by changing of the trap strength or equivalently the trap frequency and see the excitation spectrum. This will be done in Chapter 5.