CHAPTER III

EQUIVALENT PLANE STRAIN MODEL

Introduction

As mentioned in Chapter 1, the simplified three-dimensional model proposed by Hwang et al. (1975), hereafter refered to as the H.L.B. model, can be effectively used to solve three-dimensional soil medium under seismic excitation. In this study, some modification over H.L.B. model is made so as to broaden the applicability of this model to cover problems of half space under concentrated loads applied at the surface. The proposed method is of trial-and-error in nature using available analytical solutions in the literature as calibration vehicles. Several problems are analysed to demonstrate the accuracy of the method.

In the first example, the validity of the ordinary two-dimensional plane strain finite element modeling for solving the problem of rigid strip footing on elastic halfspace is demonstrated. However, when the footing becomes circular and finite in size, this modeling technique fails to give satisfactory results even with the incorporation of dashpots proposed by Hwang et al. The problem is remedied by attaching the springs to the sides of the plane strain slice to simulate the soil reaction in the third dimension. Moreover, the mass and damping properties are adjusted to produce results that will match as closely as possible with existing analytical solutions. The method proposed, to be called 'the equivalent plane strain model', is shown to be effective in solving three-dimensional soil-structure interaction problems.

Analysis of Strip Footing Using Finite Element Plane Strain Model.

Exact modeling of the infinite half space in the finite element analysis is impossible by virtue of the finite dimension of the resulting model. There are various approaches to overcome this limitation. Lysmer (1969) used simple viscous elements at the finite boundary to absorb the waves which must propagate through the outer truncated infinite media, thereby preventing the waves to reflect from the boundary. Waas (1972), Kausel et al. (1975) and some other researchers adopted the same idea of nonreflecting boundary. Although these approaches are based on some simplifying assumptions, they are quite attractive owing to their simplicity, and the results obtained are accurate enough for most practical purposes.

Judging from the degree of uncertainties of real properties and uncertainties involved in defining forces such as those caused by earthquake excitation, it is deemed sufficient to use the standard viscous boundary proposed by Lysmer (1969) throughout the study. But it should be noted that other boundary treatments can be applied as well.

Fig. 3.1a. represents a slice of the two-dimensional plane strain model of L unit width. The depth and width of the finite soil medium are H and 2B respectively. A rigid strip footing of 2b unit width rests symmetrically above the soil medium. In view of symmetry, analyses can be performed on the half model shown in Fig. 3.1b.

The basic concept to determine the standard viscous boundary properties is to specify the stress boundary conditions at the boundary of the model. Stress boundary conditions on surfaces AEHD, DHGC and CGFB in Fig. 3.1a are

$$\sigma = a \rho V_P \dot{u}_{(P)} \tag{3.1a}$$

$$\tau = b \rho V_S \dot{u}_{(S)} \tag{3.1b}$$

where σ and τ are the normal stress and the shearing stress respectively; $\dot{u}_{(P)}$ is the velocity of the normal to plane direction; $\dot{u}_{(S)}$ is the velocity of the parallel to plane direction; V_P is the P-wave velocity; V_S is the S-wave velocity; ρ is the mass density; and a and b are the dimensionless parameters which are equal to 1.0 for the standard viscous boundary. S-wave and P-wave velocities are defined by

$$V_S = \sqrt{\frac{G}{\rho}} \tag{3.2a}$$

$$V_P = \frac{V_S}{S} \tag{3.2b}$$

in which

$$S^2 = 0.5(1-2\nu)/(1-\nu) \tag{3.2c}$$

G is the shear modulus and ν is the Poisson's ratio.

The equations of motion of the finite element model shown in Fig. 3.1b without internal material damping is

$$[M]\{\ddot{q}\} + [K]\{q\} = \{P\} - \{d^b\} \tag{3.3}$$

where $[M]$ is the mass matrix; $[K]$ is the plane strain stiffness matrix; $\{P\}$ is the concentrated load acting on the centroid of the footing which can be separately considered as vertical and horizontal load, and $\{d^b\}$ is the boundary damping force vector resulting from the standard viscous boundary calculated from Eqs. (3.1) which can be expressed in terms of nodal force as :

$$d^b_{i(N)} = \rho V_P \dot{q}_{i(N)} A^b_i \tag{3.4a}$$

and

$$d^b_{i(T)} = \rho V_S \dot{q}_{i(T)} A^b_i \tag{3.4b}$$

in which the subscripts $_{i(N)}$ and $_{i(T)}$ indicate directions of the boundary viscous forces in the normal and tangential directions to the boundary planes respectively; A_i^b is the average boundary surface area centering at the ith node. Using the assemble scheme, vector $\{d^b\}$ can be written in the form

$$\{d^b\} = [D^b]\{\dot{q}\} \tag{3.5}$$

where each of the terms in the diagonal matrix $[D^b]$ equals to $\rho V_P A_i^b$ and $\rho V_S A_i^b$ for the degree-of-freedoms associated with the normal and tangential directions to the boundary planes respectively, and equals to zero elsewhere.

Substituting $\{d^b\}$ from Eq. (3.5) into Eq. (3.3) yields

$$[M]\{\ddot{q}\} + [D^b]\{\dot{q}\} + [K]\{q\} = \{P\} \tag{3.6}$$

To examine the validity of the plane strain model incorporated with the viscous boundaries for solving the problem of strip footing on elastic half space with prescribed by vertical and horizontal concentrated loads applied at the footing, the finite element model with the dimensions shown in Fig.3.1a and mesh discretization shown in Fig.3.1b was used. The elements are of the four-node bilinear quadrilateral type. A brief derivation of the element stiffness matrix is shown in the appendix. Ordinary lumped mass scheme, which can be easily evaluated, is used as mentioned in Chapter 2.

Fig. 3.2a and Fig.3.2b depict the vertical and horizontal non-dimensional displacement functions of the center point O as functions of the dimensionless frequency, a_0, which is $\omega b/V_s$. The analytical solutions given by Luco and Westmann (1972) are shown for comparison. It should be noted that the finite element solutions compare fairly well with the analytical results; however, the solutions are very sensitive to mesh configurations, thus it is difficult to evaluate percentages of such errors.

Analysis of Circular Footing Using Simplified Three-dimensional Model

We next investigate the validity of the simplified three-dimentional model as proposed by Hwang et al. (1975) for solving rigid massless circular footing on the three-dimensional half space under concentrated loads. Analytical solutions of circular footing with the same diameter as the width of strip footing used in the finite element model are referred to as the exact solutions. It should be noted that, to the writer's knowledge, the H.L.B. model has not been tested for the case of a point load prescribed directly on the footing. We will first briefly describe the H.L.B. model.

The basic idea conceived by Hwang et al. is that the waves propagating in the direction perpendicular to the plane should be absorbed by attaching damping elements on both sides of the slice, to prevent them from reflecting into the slice. Adopting the definitions and properties of these damping elements used by Lysmer and Kuhlemeyer (1969), the stress boundary conditions on both sides of the slice are in the same form as Eq. (3.1b); i.e.

$$\tau \quad = \quad \rho V_S \dot{u}_{(S)} \qquad (3.7)$$

Therefore, modifying Eq. (3.3) by adding the damping forces $\{d^s\}$ which act on both sides of the plane strain slice of width L (see Fig. 3.3), we have

$$L[M]\{\ddot{q}\} + L[K]\{q\} \quad = \quad L\{d^b\} - 2\{d^s\} \qquad (3.8)$$

The damping force vector $\{d^s\}$ can be written in terms of the nodal forces as :

$$d^s_{i(T)} \quad = \quad \rho V_S \dot{q}_{i(T)} A^s_i \qquad (3.9)$$

in which the subscript $_{i(T)}$ indicates tangential component of the forces at node i, and A_i^s is the average area on the side surface centering at i^{th} node. Note that both the vertical and horizontal forces are of the same form. Following the same procedure as in deriving the nodal forces for the viscous boundary elements, we have

$$\{d^s\} \quad = \quad [D^s]\{\dot{q}\} \qquad (3.10)$$

where each of the diagonal terms in the diagonal matrix $[D^s]$ equals to $\rho V_s A_i^s$ for every degree-of-freedom associated with the soil slice. Substituting $\{d^b\}$ and $\{d^s\}$ from Eq.(3.5) and Eq.(3.10) into Eq.(3.8) yields

$$[M]\{\ddot{q}\} + \left[[D^b] + \left(\frac{2}{L}\right)[D^s]\right]\{\dot{q}\} + [K]\{q\} \quad = \quad \{P\} \qquad (3.11)$$

The solutions of Eq.(3.11) using the finite element mesh shown in Fig.3.1b for the vertical and horizontal displacement functions of the center point O are shown in Fig.3.4a and 3.4b for the vertical and horizontal load cases, respectively. The finite element solutions do not agree well with the analytical results presented by Luco and Westmann (1971) which are also plotted in the same figure for comparison. The characteristics of the finite element H.L.B. solutions still resemble the plane strain model solutions but the amplitudes are much reduced. Overdamping phenomena are observed in the high-frequency range. Consequently, the solutions obtained by H.L.B. for concentrated loads are not on the safe side for design purposes. This indicates that the simplified three-dimensional model cannot capture the actual three dimensional behaviour very well. The difference in characteristics of the two models is also discussed by Luco and Hajian (1974). Some attempts were made to improve this model by varying the thickness of the plane strain medium. However, it is obvious that since the static solution of the plane strain problem is singular at the load point whereas that of the three-dimensional continuum is finite, such attempts are not possible.

Modification of the Plane Strain Model for Three-dimensional Analyses

As discussed in the last article, the simplified three-dimensional model is not applicable in case of a point load acting directly on a footing of finite size. To achieve more acceptable solutions, further modification must be conducted. Recalling an earlier work by Lysmer and Richart (1966), the half space was simply modeled as a single degree-of-freedom system only, comprising a spring and a dashpot. The validity of this concept hints that there may exist a suitable system of forces acting on the two-dimensional plane strain system to be equivalent to the real three-dimensional continuum. Therefore, an 'equivalent plane strain model' is proposed and tested to demonstrate the validity of such a simple model for analyses of three-dimensional soil-structure interaction problems.

The basic idea for modifying the plane strain model to make it capable of representing three-dimensional behavior is to invoke the condition that the total potential energy and the kinetic energy of the proposed system to be approximately the same as the real three-dimensional medium. Firstly, transverse spring elements are attached at each node on both sides of the plane strain slice as shown in Fig.3.5. The forces in these springs represent, in an approximate way, the interactive forces between the slice and the soil mass outside the slice. The spring stiffness is varied until the static displacement of the equivalent model is equal to that of the three-dimensional analytic solution. This strategy ensures that the total potential energy of the equivalent system is approximately the same as that of the three-dimensional continuum. Concerning the kinetic energy, it may be noted that the kinetic energy of the soil medium in the plane strain state should be greater than the actual value of the three-dimensional continuum whose displacement decays with distance in the transverse direction. The last quantity to be adjusted is the viscous damping force acting on both sides of the plane strain slice. Transforming the above stated idea into a mathematical expression results in the following equations of motion

$$C_m L[M]\{\ddot{q}\} + L[K]\{q\} = L\{P\} - L\{d^b\} - 2C_d\{d^s\} - 2C_s\{S^s\} \qquad (3.12)$$

One might observe that Eq. (3.12) is a modified form of Eq. (3.8), with the parameters C_m, and C_d introduced in order that the amount of mass and viscous damping forces can be adjusted, and the quantity $2C_s\{S^s\}$ results from the equivalent transverse interaction forces.

Determination of the vector $\{S^s\}$ is similar to that of the vector $\{d^b\}$. The difference is that $\{S^s\}$ depends on spring stiffness and nodal displacements rather than damping coefficient and velocities. The shearing stress occurring on the sides of the plane strain slice, area ABCD and EFGH in Fig. 3.1a is assumed to be

$$\tau = g^* G u_{(S)} \qquad (3.13)$$

where g^* is the multiplying parameter to adjust the shear modulus G. Thus, in analogy with Eq. (3.4), the nodal spring forces are

$$S^s_{i(T)} = g^* G q_{i(T)} A^s_i \qquad (3.14)$$

These nodal forces can be directly assembled to yield the force vector :

$$\{S^s\} = [S^s]\{q\} \qquad (3.15)$$

where each of the diagonal terms in the diagonal matrix $[S^s]$ equals to $g^* G A^s_i$ for every degree-of-freedom associated with the soil slice. Substituting Eq. (3.5), Eq. (3.10) and Eq. (3.15) into Eq. (3.12) leads to

$$C_m[M]\{\ddot{q}\} + \left[[D^b] + \left(\frac{2}{L}\right)C_d[D^s]\right]\{\dot{q}\} + \left[[K] + \left(\frac{2}{L}\right)C_s[S^s]\right]\{q\} = \{P\} \qquad (3.16)$$

As described in the introductory part, to apply this model, C_s is to be determined first. A numerical trial-and-error process is used. Eq. (3.16) is first solved for the static analysis, using some trial value of C_s. The center displacement is compared with the relevant analytical solution given by Luco and Westmann (1971), and C_s is then adjusted accordingly.

The next step is to determine the parameters C_m and C_d. As the first trial we set them equal to 1.0, corresponding to H.L.B. model. The finite element solutions obtained are shown in Fig. 3.6a and Fig. 3.6b together with the analytical results. Clearly there is the big discrepancy between the analytical and finite element solutions. Consequently, further adjustment on C_m and C_d must be performed.

As stated earlier that the kinetic energy of the plane strain model is always greater than that of the three-dimensional continuum, the parameter C_m should therefore be less than 1.0 Finally, careful inspection of the results in Fig. 3.6a and Fig. 3.6b leads to the conclusion that, for $C_d = 1.0$, the simplified model contains too much damping. Thus C_d must be less than 1.0 also.

By using the trial-and-error scheme within the domain $0 < C_m < 1.0$ and $0 < C_d < 1.0$, the solutions most conforming to the three-dimensional continuum analyses were obtained. The results are shown in Fig. 3.7a and Fig. 3.7b.

Parametric Studies

It is of interest to study the sensitiveness of the numerical results to the variations of C_m and C_d. With C_d kept constant while C_m varied within a small range, the solutions of the displacement functions were obtained and shown in Fig. 3.8a and Fig. 3.8b for comparison. Conversely, the solutions for C_m held constant while varying C_d are depicted in Fig. 3.9a and Fig. 3.9b. Variations of C_m and C_d were taken within 20 to 30 percent of the best values obtained earlier. It should be observed that a wide range of the coefficients C_m and C_d can be adopted without causing significant errors in the solutions.