CHAPTER V

MATRIX INVERSION

This matrix inversion is based on a solution of simultaneous linear equation by Gauss-Jordan mothod (i).

This is an climination method, it is best described by an example.

Ex. There are three equations with three unknowns.

$$\begin{array}{lllll} 2x & + 2y & + z & = & 2 \\ x & + 2y & - z & = & -1 \\ 3x & + y & -2z & = & -6 \end{array}$$

Can be reduced to

$$\begin{array}{lllll} ax & & & = & p \\ & by & & = & q \\ & & cz & = & r \end{array}$$

The problem can be reduced step by step

$$\begin{array}{lllll} 2x & + 2y & + z & = & 2 \\ & y & -3/2z & = & -2 \\ & - 2y & -7/2z & = & -9 \end{array}$$

$$\begin{aligned} 2x \quad & \quad + 4z = 6 \\ y \quad & -5/2z = -2 \\ & -13/2z = -13 \end{aligned}$$

$$\begin{aligned} 2x \quad & = -2 \\ y \quad & = 1 \\ -13/2z & = -13 \end{aligned}$$

$\therefore \quad x = -1, \quad y = 1, \quad z = 2$

General form of the equations

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots\cdots\cdots\cdots a_{1n}X_n &= b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots\cdots\cdots\cdots a_{2n}X_n &= b_2 \\ a_{n1}X_1 + a_{n2}X_2 + \cdots\cdots\cdots\cdots a_{nn}X_n &= b_n \end{aligned}$$

<u>Coefficient matrix</u>.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots\cdots & a_{nn} \end{vmatrix}$$

<u>Augmented matrix</u> is the coefficient matrix with the right hand side of the equation included.

$$\begin{pmatrix} a_{11} & a_{12} \cdots\cdots\cdots\cdots & a_{1n} & b_1 \\ a_{21} & a_{22} \cdots\cdots\cdots\cdots & a_{2n} & b_2 \\ a_{n1} & a_{n2} \cdots\cdots\cdots\cdots & a_{nn} & b_n \end{pmatrix}$$

The previous problem can be represented in terms of the augmented matrix, that is

$$\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & a_{13} & a_{14} & 2 & 2 & 1 & 2 \\ a_{21} & a_{22} & a_{23} & a_{24} & 1 & 2 & -1 & -1 \\ a_{31} & a_{32} & a_{33} & a_{34} & 3 & 1 & -2 & -6 \end{array}\right)$$

Where the numerical values are shown alongside their symbolic names.

To eliminate a_{21}, a factor was selected which, when multiplied onto the first row and added to the second, term by term, the second row becomes

$$a'_{21} = a_{21} + f.a_{11}$$

$$a'_{22} = a_{22} + f.a_{12}$$

$$a'_{23} = a_{23} + f.a13$$

$$a'_{24} = a_{24} + f.a_{14}$$

The primes do not indicate a seperate matrix. They show how the original matrix becomes transformed by the elimination procedure. That is, during the course of the computation, each element in the original matrix is replaced by a primed element according to a scheme.

$$f = -a_{21}/a_{12} \tag{5-1}$$

<u>Solution</u>.

$$\begin{bmatrix} 2 & 2 & 1 & 2 \\ 1 & 2 & -1 & -1 \\ 3 & 1 & -2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 2 \\ 0 & 1 & -3/2 & -2 \\ 0 & -2 & -7/2 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & +4 & 6 \\ 0 & 1 & -3/2 & -2 \\ 0 & 0 & -13/2 & -13 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -13/2 & -13 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

For accuracy purposes, the max. element in each column should be selected as a pivot. Since the choice of pivot rows is unpredictable, the program will have to keep track on two things.

1. To ensure will not be used more than once for pivoting, an array Ck(n) will be maintained so that a 1 in. Ck(n) indicates that row n has been used, a zero, that it has not.

2. Sequence in which pivot rows occur will be noted in array LOC. If LOC(I) = J, this indicates that in the elimination on the I^{th} column, the max. eligible pivot was found in row J.

<u>EX.</u>

<u>Stage 1</u>

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & -1 & 1 & 2 \end{array}\right)$$

	I	Ck(I)	LOC(I)
F = -1/2	1	0	3
F = -1/2	2	0	-
	3	I	-

Stage 2

$$\left(\begin{array}{ccc|c} 0 & 3/2 & 3/2 & 0 \\ 0 & 3/2 & 5/2 & 2 \\ 2 & -1 & 1 & 2 \end{array}\right)$$

F = -1

F = 2/3

I	Ck(I)	LOC(I)
1	I	3
2	0	1
3	I	-

Stage 3

$$\left(\begin{array}{ccc|c} 0 & 3/2 & 3/2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 2 & 2 \end{array}\right)$$

F = -3/2

F = -2

I	Ck(I)	LOC(I)
1	I	3
2	I	1
3	I	2

Final

$$\left(\begin{array}{ccc|c} 0 & 3/2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & -2 \end{array}\right)$$

X(I)

X(1) = -2/2 = -1

X(2) = $-3/\frac{3}{2}$ = -2

X(3) = s/1 = 2

Hence

$$\begin{pmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Let X be an inverse matrix of a square matrix A.

Hence $\quad A\,X = I$

where I is a diagonal matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Because of the way matrix A is multiplied on to X, that is, on to a column X at a time, it can be written

$$A \begin{pmatrix} X_{11} \\ X_{21} \\ X_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, A \begin{pmatrix} X_{12} \\ X_{22} \\ X_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, A \begin{pmatrix} X_{13} \\ X_{23} \\ X_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Each equation can be solved as in the case of simultaneous linear equations. The results would be

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & X_{11} \\ 0 & 1 & 0 & X_{21} \\ 0 & 0 & 1 & X_{31} \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & X_{12} \\ 0 & 1 & 0 & X_{22} \\ 0 & 0 & 1 & X_{32} \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & X_{13} \\ 0 & 1 & 0 & X_{23} \\ 0 & 0 & 1 & X_{33} \end{array}\right)$$

This can be done simultaneously, the three augmented matrix are put together.

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

The final result would be

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & X_{11} & X_{12} & X_{13} \\ 0 & 1 & 0 & X_{21} & X_{22} & X_{23} \\ 0 & 0 & 1 & X_{31} & X_{32} & X_{33} \end{array}\right)$$

Two arrays are introduced when accuracy is needed.

1. If LOC(I) = J, it means that at column I, the pivoting value is at row J.

2. LAC(J) = I, it means that at row J, the pivoting value is at column I.

EX.

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 4 & 2 & 1 & 1 \\ 3 & 4 & 1 & 0 \end{array}\right) = \left(\begin{array}{ccc|cc} 0 & 5/2 & 7/4 & -1/4 & 1 \\ 1 & 1/2 & 1/4 & 1/4 & 0 \\ 0 & 5/2 & 1/4 & -3/4 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 0 & 1 & 7/10 & -1/10 & 2/5 & 0 \\ 1 & 0 & -1/10 & 3/10 & -1/5 & 0 \\ 0 & 0 & -3/2 & -1/2 & -1 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & -1/3 & -1/5 & 7/15 \\ 1 & 0 & 0 & +1/3 & -2/15 & -1/15 \\ 0 & 0 & 1 & 1/3 & 2/3 & -2/3 \end{array}\right)$$

Direct method

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 1 \\ 3 & 4 & 1 \end{pmatrix} \quad \begin{bmatrix} -2/15 & 1/3 & -1/15 \\ -1/15 & -1/3 & 7/15 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}^{-1}$$

with

LOC(1) = 2 ; LAC(2) = 1

LOC(2) = 1   LAC(1) = 2

LOC(3) = 3   LAC(3) = 3

It is learned that the results obtained from the first method must be interchanged rows, 1 and 2 first and then interchange column, 1 and 2, then the right results is obtained.

LOC(I) and LAC(J) can be used in finding the right matrix.

LOC(I) = J, it indicates that the right row 1 is at row J, and LAC(I) = J, indicates that the right column I is at column J. Only NxN+1 matrix is needed in this matrix inversion programme. One column of unit matrix on which contains 1 in a pivoting row is put in the $n + 1^{th}$ column. When this cycle of calculation is finished, the $n+1^{th}$ column will be transferred to the pivoting column. Then the next pivoting row will be inspected in the next column, and a column of unit matrix contained 1 at the new pivoting row will be put in a $n + 1^{th}$ column.

Admittance matrix input is excluded the first row and column which are nodal and mutual components of a slack busbar. Columns are considered one after another, begining from the 1st to the n^{th} column. At each column consideration, max. row element is found in 2., and indicators are set. The $n+1^{th}$ column is set zero except the pivoting row. In step 5 pivoting row is devided by the pivoting element. At this stage the elimination is performed. All rows except the pivoting row are considered one at a time.

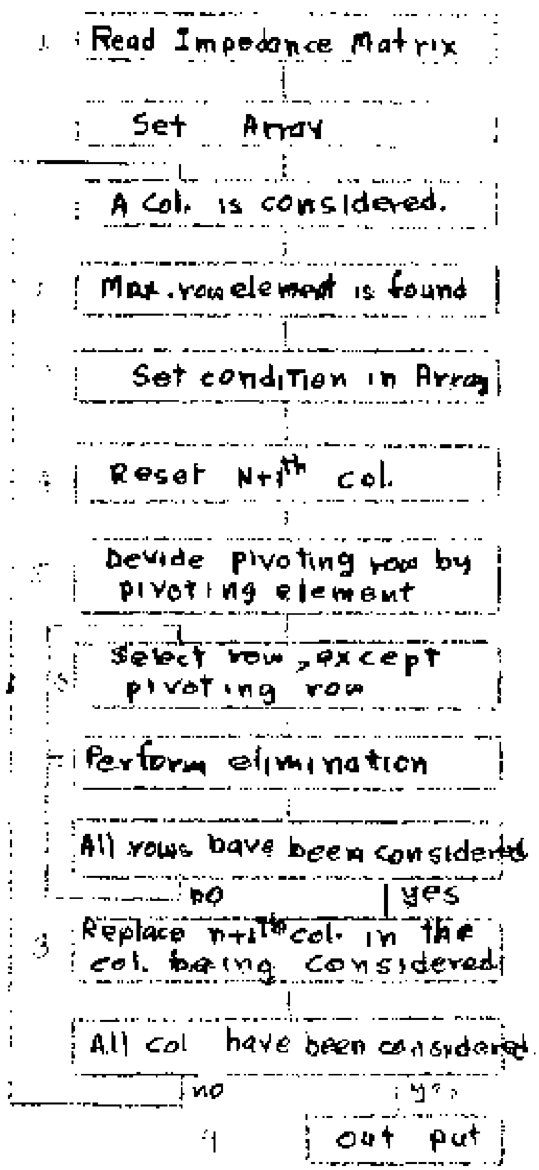

Figure 8 Matrix Inversion

The elements of the selected row are multiplied by a constant obtained by the equation (5-1) and substracted by the same column element of the pivoting row. $n+1^{th}$ column is put in places of pivoting column when all rows have considered. In step 8 the next column will be considered. When all column have been performed, the invert matrix will be at the 1^{th} to n^{th} column. The actual invert matrix will be rearranged by means of the array indicators as the previous description. Then the output is typed and punched.